FIXED POINT THEOREMS IN ANALYSIS

by

© EDWARD RICHARD GRAFF

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "FIXED POINT THEOREMS IN ANALYSIS", submitted by EDWARD RICHARD GRAFF in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis consists of a study of some of the classical fixed point theorems of Functional Analysis together with their later generalizations.

In Chapter I, we consider the Banach Contraction Mapping theorem and its later extensions by M. Edelstein, E. Rakotch and others.

Chapter II consists of a development of the simplicial homology theory needed to define the Brouwer degree of a continuous mapping between n-cells, and to establish the Brouwer fixed point theorem.

In Chapter III we consider the most important generalizations of the Brouwer theorem; these include the fixed point theorems of Lefschetz, Schauder, and Tychouov.

In the last chapter some applications of fixed point theorems are presented.

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CHAPTER I

THE CONTRACTION MAPPING THEOREM

An important part of fixed point theory is concerned with the existence of fixed points of functions on metric spaces which in some sense decrease the distance between points. The simplest of these theorems, and the point of departure for many theorems dealing with "contraction", "contractive", or "nonexpansive" mappings and their generalizations, is the Contraction Mapping Theorem of S. Banach.

This chapter is devoted to the Banach theorem and some of its generalizations; examples of applications of the theorems will be found in Chapter IV.

<u>I.1. Definition</u>: Let (X,d) be a metric space; a self-mapping f of X is said to be a <u>contraction</u> if there is a number a, $0 \leq a < 1$, such that for any x,y in X,

$$d(f(x),f(y)) \leq ad(x,y)$$

<u>I.2. Theorem</u> (Banach [1]) If f is a contraction defined on a complete metric space (X,d), then f has a unique fixed point x' in X. Furthermore, for all x in X, we have

$$x' = \lim_n f^n(x) \quad ,$$

where $f^n(x) = f(f^{n-1}(x))$, $n = 1,2,\dots$.

<u>Proof</u>: Let x_0 be an arbitrary point of X , and define a sequence $\{x_n\}$ in X by $x_1 = f(x_0)$,..., $x_k = f(x_{k-1}) = f^k(x_0)$,... . We shall show that $\{x_n\}$ is a Cauchy sequence. In fact, for integers m,n, with $m \geq n$,

$$\begin{aligned} d(x_n, x_m) = d(f^n(x_0), f^m(x_0)) &\leq a^n d(x_0, x_{m-n}) \\ &\leq a^n[d(x_0,x_1) + d(x_1,x_2) + \dots + d(x_{m-n-1},x_{m-n})] \\ &\leq a^n d(x_0,x_1)[1 + a + a^2 + \dots + a^{m-n-1}] \\ &< a^n d(x_0,x_1)(1-a)^{-1} \end{aligned}$$

Since a is less than 1 , this last quantity is arbitrarily small for sufficiently large n , and hence the sequence is Cauchy. By completeness of X , there is a point x' in X such that $\lim_n x_n = x'$. A contraction mapping is clearly continuous, so we have

$$f(x') = f(\lim_n x_n) = \lim_n f(x_n) = \lim_n x_{n+1} = x' \quad ,$$

and x' is a fixed point of f .

If y is any point of X such that $f(y) = y$, then $d(x',y) = d(f(x'),f(y)) \leq ad(x,y)$, and we must have $d(x',y) = 0$. That is, $x' = y$ and uniqueness of the fixed point is shown.

I.3. Remarks: (i) The contraction condition $d(f(x),f(y)) \leq ad(x,y)$, $a < 1$, can not, in general, be relaxed to

$$d(f(x),f(y)) < d(x,y) \quad , \quad x \neq y$$

as the following example shows. Let X be the set of real numbers $x \geq 1$, with $d(x,y) = |x-y|$, and let $f(x) = x + 1/x$. Then X is complete, and

$$d(f(x),f(y)) = |x - y + (y - x)/xy| < |x - y| = d(x,y) \quad ,$$

but f has no fixed point. Many of the later generalizations of Banach's theorem are concerned with weakening the contraction condition.

(ii) Alternate proofs of the Contraction Mapping Theorem, using the Cantor Intersection Theorem, have been given by I. I. Kolodner [1], and D. W. Boyd and J. S. W. Wong [1]. Briefly, the proof of Boyd and Wong is as follows: since f is a contraction, the mapping $g(x) = d(x,f(x))$ is continuous and satisfies $g(f^n(x)) \to 0$, as $n \to \infty$, for any x in X. Consequently, the sets

$$C(m) = \{x \text{ in } X : d(x,f(x)) \leq 1/m\} \quad , \quad m = 1,2,...$$

are closed and nonempty; also, $f(C(m))$ is contained in $C(m)$. Further, if x,y belong to $C(m)$

$$\begin{aligned} d(x,y) &\leq d(x,f(x)) + d(f(x),f(y)) + d(y,f(y)) \\ &\leq 2/m + ad(x,y) \end{aligned}$$

so that $\operatorname{diam} C(m) \leq 2/m(1 - a)$. Thus the conditions

of Cantor's theorem are satisfied, and the intersection of the $C(m)$ consists of a single point. This point is the unique fixed point of f.

(iii) Some results have also been obtained on the converse of the Contraction Mapping Theorem, of which the following theorem is representative:

<u>Theorem</u> (C. Bessaga [1]) Let X be an abstract set, and let T be a self-mapping of X. If there is a unique point $x^* \varepsilon X$ such that $T^k(x^*) = x^*$, $k = 1,2,\ldots,$ then for each $a \varepsilon [0,1)$, there is a complete metric d on X such that

$$d(T(x),T(y)) \leq ad(x,y),$$

for all x,y in X.

Further results relative to converses of the Contraction Mapping Theorem may be found in Janos [1], and Wong [1].

(iv) It will be useful to note here that if an iterate f^n of a mapping has a unique fixed point x, i.e., if x is a unique periodic point of f, then x is a unique fixed point of f. For, $f(x) = f(f^n(x)) = f^n(f(x))$ so that $f(x)$ is a fixed point of f^n; by uniqueness, $f(x) = x$. That x is a unique fixed point of f follows, of course, from the

fact that any fixed point of f is also a fixed point of any iterate of f.

By virtue of this remark, the following theorem of J. Weissinger is seen to be equivalent to Theorem I.2.

Theorem (J. Weissinger [1]) Let f be a self-mapping of a complete metric space (X,d), and suppose $\{a_n\}$ is a sequence of real numbers such that

(i) $d(f^n(x),f^n(y)) \leq a_n d(x,y)$, for all x,y in X; $n = 1,2,\ldots.$

(ii) $\sum a_n$ is convergent.

Then f has a unique fixed point x' which satisfies

$$x' = \lim_n f^n(x) \quad \text{for each } x \text{ in } X .$$

Also, in view of the above remark a slight generalization of Banach's theorem is obtained by requiring only that an iterate of f, and not f itself, be a contraction. That such mappings f exist is shown by the following example (Chu and Diaz [1]):
Let X be the set of real numbers with the usual metric, and let H be a Hamel basis for the reals which contains 1 and π (i.e., H is a set of real numbers, linearly independent over the field of rationals, such that any nonzero real number is a unique finite linear

combination of elements of H, with rational coefficients). Define f on X by $f(0) = 0$, $f(1) = \pi/2$, $f(\pi) = 1/2$, $f(h) = h/2$ for h in $H - \{1,\pi\}$, and for any nonzero real number $x = \sum r_i h_i$, $f(x) = \sum r_i f(h_i)$. Then, f^2 is a contraction, since for any real $y, f(f(y)) = y/4$. Furthermore, f is additive, $f(x + y) = f(x) + f(y)$, so that in order to be continuous f must also satisfy $f(x) = f(1)\ x$ for all x. But $f(1) = \pi/2$ so that $f(\pi) = f(1)\pi = \pi^2/2$, which is a contradiction. Thus, f is not continuous and can not, therefore, be a contraction.

We now turn our attention to some extensions of the Contraction Mapping Theorem. The main results are due to M. Edelstein (Corollary I.6., Theorem I.9.) and A. Meir (Theorem I.10.).

<u>I.4. Definition</u>: A mapping f of a metric space (X,d) into itself is said to be

(i) <u>contractive</u> if for all x,y, in X

$$d(f(x),f(y)) < d(x,y)$$

(ii) <u>e-contractive</u>, for some positive number e, if for all x,y in X, $0 < d(x,y) < e$ implies $d(f(x),f(y)) < d(x,y)$

<u>I.5. Theorem</u> (Edelstein [1]) Let (X,d) be a metric space, f an e-contractive mapping on X. If there is a point x^* in X such that the sequence of iterates $f^n(x^*)$ has a subsequence convergent to a point w in X, then w is a periodic point of f; i.e., for some positive integer $k, f^k(w) = w$.

<u>Proof</u>: Suppose $\{f^{n_i}(x^*)\}$ is a subsequence convergent to w; we shall write $x_i = f^{n_i}(x^*)$. Let N be an integer such that $i \geq N$ implies $d(x_i, w) < e/4$, and for fixed $i \geq N$, let $k = n_{i+1} - n_i$. Since f is an e-contraction, so is any iterate of f, and we have

$$d(x_{i+1}, f^k(w)) = d(f^k(x_i), f^k(w)) < d(x_i, w) < e/4 .$$

Consequently,

$$d(w, f^k(w)) \leq d(w, x_{i+1}) + d(x_{i+1}, f^k(w)) < e/2 .$$

Suppose $v = f^k(w) \neq w$; then by e-contractiveness, $d(f(w), f(v)) < d(w,v)$; or, $d(f(w),f(v))/d(w,v) < 1$. The function $h(x,y) = d(f(x),f(y))/d(x,y)$ is clearly continuous off the diagonal of X, hence is continuous at (w,v). Thus there is a $d > 0$ and an M in $(0,1)$ such that, for $x \neq y$,

$$d(x,w) < d \text{ and } d(y,v) < d \text{ implies } d(f(x),f(y)) < Md(x,y) .$$

Also, $\lim_r f^k(x_r) = f^k(w) = v$, so that for some $N' \geq N$, $r \geq N'$ implies

$$d(x_r, w) < d \quad \text{and} \quad d(f^k(x_r), v) < d .$$

Thus, for $r \geq N'$,

$$d(f(x_r),f(f^k(x_r))) < Md(x_r,f^k(x_r)) ;$$

and

$$d(x_r,f^k(x_r)) \leq d(x_r,w) + d(w,f^k(w)) + d(f^k(w),f^k(x_r))$$

$$< e/4 + e/2 + e/4 = e .$$

From these inequalities follows

$$d(f(x_r),f(f^k(x_r))) < Md(x_r,f^k(x_r)) < Me < e .$$

Hence by e-contractiveness of f and its iterates, we have, for any integers $p > 0$ and $r \geq N'$,

$$d(f^p(x_r),f^p(f^k(x_r))) < Md(x_r,f^k(x_r)) .$$

Setting $p = n_{r+1} - n_r$, this last expression becomes

$$d(x_{r+1},f^k(x_{r+1})) < Md(x_r,f^k(x_r)) .$$

Thus, for $s \geq r$,

$$d(x_s,f^k(x_s)) < M^{s-r}d(x_r,f^k(x_r)) < M^{s-r}e .$$

But

$$d(w,v) \leq d(w,x_s) + d(x_s,f^k(x_s)) + d(f^k(x_s),v)$$

$$\to 0 \quad \text{as} \quad s \to \infty .$$

Thus, we conclude that $w = v = f^k(w)$, and the theorem is proved.

<u>I.6. Corollary</u>: Let (X,d) be a metric space, f a contractive self-mapping of X such that for some x^* in X the sequence of iterates has a subsequence which converges to a point w in X. Then w is a fixed

point for f and is unique.

Proof: According to the theorem, there is a positive integer k such that $f^k(w) = w$. Suppose $v = f(w) \neq w$. Then $f^k(v) = f^{k+1}(w) = f(w) = v$, and we have

$$d(w,v) = d(f^k(w),f^k(v)) < d(w,v) .$$

Since this is impossible, we must have $v = w$, and w is a fixed point of f. Similarly, if $f(u) = u$ and $u \neq w$,

$$d(w,u) = d(f(w),f(u)) < d(w,u) ,$$

so that uniqueness follows.

I.7. Definition: A mapping f of a metric space (X,d) into itself is said to be locally contractive if for each x in X there exist $e = e(x) > 0$ and $a = a(x)$, $0 \leq a < 1$, such that

$$p,q \text{ in } S(x,e) \text{ implies } d(f(p),f(q)) < ad(p,q)$$

$$(S(x,e) = \{y : d(x,y) < e\}) .$$

If e and a are independent of x, f is said to be (e,a) - uniformly contractive.

I.8. Definition: (i) A metric space (X,d) is said to be e-chainable, $e > 0$, if, for any two points x,y in X, there is a finite set of points $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{i-1},x_i) < e$,

$i = 1,2,\ldots\ldots,n$. Such a set of points will be called an e-chain.

(ii) The chain length, $L(C)$, of an e-chain $C : x = x_0,x_1,\ldots,x_n = y$ is defined by

$$L(C) = d(x_1,x_0) + \ldots + d(x_n,x_{n-1})$$

I.9. Theorem (Edelstein [2]) An (e,a) - uniformly locally contractive mapping f on a complete, e-chainable metric space (X,d) possesses a unique fixed point.

Proof: Let x^* in X be arbitrary; the sequence of iterates $f^n(x^*)$ is a Cauchy sequence. To see this, consider an e-chain $x^* = x_0,x_1,\ldots,x_n = f(x^*)$. By definition, $d(x_{i-1},x_i) < e$, $i = 1,\ldots,n$; i.e., x_{i-1} and x_i belong to $S(x_i,e)$, which implies

$$d(f(x_i),f(x_{i-1})) < ad(x_i,x_{i-1}) < ae \quad .$$

Induction on the iterates of f gives

$$d(f^m(x_{i-1}),f^m(x_i)) < a^m e \quad , \quad m = 1,2,\ldots$$

Thus,

$$d(f^m(x^*),f^{m+1}(x^*)) = d(f^m(x^*),f^m(f(x^*)))$$

$$\leq \textstyle\sum_{i=1}^{n} d(f^m(x_{i-1}),f^m(x_i)) < a^m ne$$

Now, for positive integers p,q , with $p < q$,

$$d(f^p(x^*),f^q(x^*)) \leq d(f^p(x^*),f^{p+1}(x^*)) + \ldots + d(f^{q-1}(x^*),f^q(x^*))$$
$$< ne(a^p + \ldots + a^{q-1})$$
$$< nea^p/(1-a) \to 0 \quad \text{as} \quad p \to \infty ,$$

and $\{f^n(x^*)\}$ is indeed Cauchy. Since X is complete, there is a w in X such that $\lim_n f^n(x^*) = w$.

It is easy to see that a local contraction is continuous, so we have

$$f(w) = f(\lim_n f^n(x^*)) = \lim_n f(f^n(x^*)) = \lim_n f^{n+1}(x^*) = w$$

and w is a fixed point of f .

Next, suppose y in x is such that $f(y) = y$.

Let $w = y_0,\ldots,y_k = y$ be an e-chain linking w and y .

As above we have

$$d(f^j(w),f^j(y)) \leq \sum_{i=1}^k d(f^j(y_{i-1}),f^j(y_i))$$
$$< ka^j e \to 0 \quad \text{as} \quad j \to \infty$$

Thus $w = f^j(w) = f^j(y) = y$, which shows uniqueness.

I.10. <u>Theorem</u> (A. Meir [1]) Let (X,d) be a complete metric space, and suppose f is a self-mapping of X which possesses the following property:

> For each number $e > 0$ there is a number $e^* > 0$ such that, for x,y in X ,
>
> $d(x,y) > e$ implies $d(f(x),f(y)) < d(x,y) - e^*$.

Then f has a unique fixed point which, for x arbitrary in X , is the limit of the sequence $\{f^n(x)\}$.

Proof: Let x be an arbitrary point of X and define $x_n = f^n(x)$, $c_n = d(x_n, x_{n+1})$, $n = 1,2,\ldots$. The sequence $\{c_n\}$ decreases to zero; to see this, note that

$$c_{n+1} = d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) < d(x_n, x_{n+1}) - e^*_n$$
$$< d(x_n, x_{n+1}) = c_n \quad , \quad n = 1,2,\ldots$$

where $0 < e_n < d(x_n, x_{n+1})$ (if $d(x_n, x_{n+1}) = 0$, then x_n is a fixed point of f , and there is nothing to prove). Thus, $c = \lim_n c_n$ exists; if $c > 0$, then $c_n \geq c > 0$ and, by the assumed property of f , $c_{n+1} < c_n - c^*$. Letting $n \to \infty$, we get $c^* \leq 0$, which is a contradiction; therefore $c = \lim_n c_n = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose not; then for some $e > 0$ and arbitrarily large (suitable) values of m and n ,

$$d(x_m, x_n) > e \quad .$$

But for all m and n ,

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n)$$
$$\leq c_m + d(x_m, x_n) - e^* + c_n$$

from which

$$e^* \leq c_m + c_n \to 0 \quad .$$

This contradicts $e^* > 0$, and we conclude $\{x_n\}$ is Cauchy.

Thus, by completeness of X , $\lim_n f^n(x) = w$ exists, and by continuity of f , w is a fixed point of f . If $y \neq w$ is a fixed point of f , then we have

$$d(w,y) = d(f(w),f(y)) \leq d(w,y) - (1/2d(w,y))^* < d(w,y)$$

which is impossible, and uniqueness follows.

Meir's theorem has as corollaries several earlier results, in which the contraction condition

$$d(f(x),f(y)) \leq ad(x,y) \quad , \quad 0 < a < 1$$

is replaced by a condition of the form

$$d(f(x),f(y)) \leq G(d(x,y)) \quad ;$$

where G is a non-negative real function on $[0,\infty)$ For these theorems, we state the following definitions:

<u>I.11. Definition</u>: An extended real valued function f defined on a topological space X is <u>upper semi-continuous</u> if

(i) for each x in X , $-\infty \leq f(x) < \infty$; and

(ii) for each real number r , the set $\{x \text{ in } X : f(x) < r\}$ is open in X .

<u>I.12. Definition</u>: A metric space (X,d) is <u>metrically convex</u> if for any two points x and y of X , there is a point z such that

$$d(x,y) = d(x,z) + d(z,y)$$

Let F denote the class of monotonically decreasing functions $a(x)$ defined on $[0,\infty)$ with range in $[0,1)$. For a in F and x,y in a metric space (X,d), we shall write $a(x,y)$ for $a(d(x,y))$.

I.13. Corollary (Rakotch [2]): Let (X,d) be a complete metric space, and suppose f is a self-mapping of X such that

$$d(f(x),f(y)) \leq a(x,y)d(x,y)$$

for all x,y in X and some a in F. Then f has a unique fixed point.

Proof: Let $e > 0$; for $d(x,y) > e$, $a(x,y) \leq a(e) < 1$ since a is in F. Thus

$$\begin{aligned} d(f(x),f(y)) &\leq a(x,y)d(x,y) \leq a(e)d(x,y) \\ &= d(x,y) - (1-a(e))d(x,y) \\ &< d(x,y) - (1-a(e))e \,. \end{aligned}$$

With $e^* = (1-a(e))e$, the condition of the theorem is satisfied, and f has a unique fixed point.

I.14. Corollary (Boyd and Wong [2]) Let G be an upper semi-continuous function on $[0,\infty)$ such that

$$G(t) < t \quad \text{for all } t > 0$$

$$\liminf_{t \to \infty} (t-G(t)) = s > 0$$

If a self-mapping f of a complete metric space (X,d) satisfies

$$d(f(x),f(y)) \le G(d(x,y)) \quad , \quad x,y \text{ in } X \quad , x \ne y$$

then f has a unique fixed point.

Proof: Since $\liminf_{t \to \infty} (t-G(t)) = s > 0$, it follows that

$$G(t) < t-s/2 \quad , \quad t > t_0$$

for some t_0 . By upper semi-continuity and the fact that $G(t) - t < 0$, we have for each positive e ,

$$\max_{e \le t \le t_0} (G(t) - t) = - d(e) = - d < 0 \quad .$$

Let $e^* = \min(d/2,s/2)$; then for $d(x,y) > e$

$$d(f(x),f(y)) \le G(d(x,y)) < d(x,y) - e^*$$

and the theorem applies.

I.15. Lemma: If (X,d) is a complete metrically convex metric space, then for any a in $(0,1)$ and any x,y in X , there exists z in X such that

$$d(x,z) = ad(x,y)$$

$$d(z,y) = (1-a)d(x,y)$$

Proof: See Blumenthal [1] p.41, Theorem 14.1.

I.16. Corollary (Boyd and Wong [2]) Suppose that (X,d) is a complete metrically convex metric space and that f is a self-mapping of X which satisfies

$$d(f(x),f(y)) \leq G(d(x,y)) \quad , \qquad x \neq y$$

where $G : [0,\infty) \to [0,\infty)$ satisfies $G(t) < t$ for all $t > 0$.

Then f has a unique fixed point, x^*, and for each x in X, $f^n(x)$ converges to x^*.

Proof: Let $e > 0$; if $d = d(x,y) > e$ then it follows from the preceding lemma that there is a z in X such that

$$d(x,z) = e \quad \text{and} \quad d(z,y) = d - e \quad .$$

Thus, for $d(x,y) > e$, we have

$$\begin{aligned} d(f(x),f(y)) &\leq d(f(x),f(z)) + d(f(z),f(y)) \\ &\leq G(d(x,z)) + G(d(z,y)) \\ &= G(e) + G(d-e) \\ &< G(e) + d - e = d(x,y) - (e-G(e)) \quad . \end{aligned}$$

With $e^* = e - G(e)$, the theorem can be applied and Corollary I.16., follows.

Using the result of Corollary I.13., Rakotch has proved the following generalization of Edelstein's theorem, Theorem I.9.:

I.17. Theorem (Rakotch [1]) Let (X,d) be a complete e-chainable metric space, and let f be a self-mapping of X such that for some a in F (as defined on Page 14),

$$d(x,y) < e \quad \text{implies} \quad d(f(x),f(y)) \leq a(x,y)d(x,y) \quad .$$

Then f has a unique fixed point.

<u>Proof</u>: For x,y in X, define

$$p(x,y) = \inf\{L(C) : C \text{ is an e-chain linking } x \text{ and } y\}.$$

Then p is clearly a metric on X ; furthermore, for any x,y in X, $d(x,y) \leq L(C)$, where C is an e-chain linking x and y, so that $d(x,y) \leq p(x,y)$. Hence $\{x_n\}$ Cauchy in (X,p) implies $\{x_n\}$ is a Cauchy sequence in (X,d) and therefore converges, say to x. For an arbitrary d in $(0,e)$ and n sufficiently large, $d(x,x_n) < d$, so x,x_n is an e-chain and we have

$$p(x,x_n) < d \quad ;$$

i.e., (X,p) is complete.

Consequently, by Corollary I.13., it now suffices to show that there is an a^* in F for which

$$p(f(x),f(y)) \leq a^*(x,y)p(x,y)$$

For $d(x,y) < e$, we have $d(x,y) = p(x,y)$ so, by hypothesis, we may set $a^*(t) = a(t)$ whenever $t < e$. We now show that for $d(x,y) \geq e$, there is a constant b in $(0,1)$ such that

$$p(f(x),f(y)) \leq bp(x,y) \quad ,$$

and define $a^*(t) = b$ whenever $t \geq e$. The function a^* is then clearly in F and satisfies the above condition.

Suppose $d(x,y) \geq e$, and let

$$C : x = y_0, y_1, \ldots, y_n = y$$

be an e-chain linking x and y , with the property that no subset of C is an e-chain linking x and y (such an e-chain is said to be strict). Let M denote the set

$$\{Y_i = (y_i, y_{i-1}) : d(y_i, y_{i-1}) < e/2\}$$

Now, no two consecutive pairs $(y_k, y_{k-1}), (y_{k+1}, y_k)$ belong to M , for if so,

$$d(y_{k-1}, y_{k+1}) \leq d(y_k, y_{k-1}) + d(y_{k+1}, y_k) < e/2 + e/2 = e$$

and C contains the e-chain $x = y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n = y$, contradicting the strictness of C . Thus if m is the number of elements of M , we have $m \leq n - m + 1$, while also $m \leq n - 1$ ($d(x,y) \geq e$ implies $n > 1$) ; consequently, $m \leq 2n/3$

Thus,

$$\sum_{Y_i \varepsilon M} d(y_i, y_{i-1}) < 2L(C)/3$$

It follows from this that for every x,y in X and every strict e-chain $C : y_0, \ldots, y_n$ between x and y ,

$$p(f(x), f(y)) \leq \sum_{i=1}^{n} d(f(y_i), f(y_{i-1}))$$

$$= \sum_{Y_i \varepsilon M} d(f(y_i), f(y_{i-1})) + \sum_{Y_i \notin M} d(f(y_i), f(y_{i-1}))$$

$$\leq \sum_{Y_i \varepsilon M} d(y_i, y_{i-1}) + \sum_{Y_i \notin M} a(e/2) d(y_i, y_{i-1})$$

$$= a(e/2) \sum_{i=1}^{n} d(y_i, y_{i-1}) + (1-a(e/2)) \sum_{Y_i \varepsilon M} d(y_i, y_{i-1})$$

$$< a(e/2) L(C) + (1-a(e/2))(2L(C)/3)$$

$$= (a(e/2)+2) L(C)/3 \quad .$$

It is clear that in the definition of $p(x,y)$ it suffices to take the inf over the set of $L(C)$ such that C is a strict e-chain linking x and y.

Thus, we have, for every x,y in X with $d(x,y) \geq e$,

$$p(x,y) \leq bp(x,y)$$

where

$$b = (a(e/2) + 2)/3 < 1$$

which was to be shown.

The Contraction Mapping Theorem has also been generalized to uniform spaces, with results analogous to the preceding metric space theorems being established by Edelstein [3], Kammerer and Kasriel [1], Knill [1], and S. A. Naimpally [1].

CHAPTER II

THE BROUWER FIXED POINT THEOREM

In a series of papers written between 1910 and 1913, L. E. J. Brouwer defined the degree of a continuous mapping of one n-dimensional manifold into another, and used this to prove the following by now classical fixed point theorem: (Brouwer [1],[2],[3]).

B1. A continuous self-mapping of an n-cell has a fixed point.

This result represented the first major step in the theory of fixed points of mappings, and served as a starting point for most subsequent investigations.

In this chapter we develop the aspects of simplicial homology theory necessary to define the Brouwer degree, and establish the properties of the degree used in proving B1. This homology theory will also be used later in discussing extensions and generalizations of Brouwer's theorems.

We also present the elementary proof, due to Knaster, Kuratowski, and Mazurkiewicz, of the following equivalent form of B1:

B2. A continuous self-mapping of an n-simplex possesses a fixed point.

II.1. Definition: (i) A set $A = \{a_0, \ldots, a_k\}$ of $k + 1$ vectors in n-dimensional Euclidean space E^n is said to be pointwise independent (or affine independent) if the k vectors $a_1 - a_0, \ldots, a_k - a_0$ are linearly independent.

(ii) If $A = \{a_0, \ldots, a_k\}$ is pointwise independent and if for a vector x in E^n there exist real numbers $c_0, \ldots, c_k$ such that

$$x = c_0 a_0 + \ldots + c_k a_k \quad , \qquad c_0 + \ldots + c_k = 1 \ ,$$

then the numbers $c_0, \ldots, c_k$ are called the barycentric coordinates of x with respect to A .

II.2. Remarks: (i) We note that the vector a_0 does not occupy a special position in the above definition of pointwise independence; for, if $a_1 - a_0, \ldots, a_k - a_0$ are linearly independent, then so are the vectors $a_0 - a_i, \ldots, a_{i-1} - a_i, a_{i+1} - a_i, \ldots, a_k - a_i$ for any choice of $i = 1, \ldots, k$.

For example, in the case $i = k$, we have

$$0 = \sum_{j=0}^{k-1} c_j(a_j - a_k) = \sum_{j=0}^{k-1} [c_j(a_j - a_0) - c_j(a_k - a_0)]$$

implies $c_j = 0$, $j = 0,\ldots,k-1$, since $a_1 - a_0,\ldots,a_k - a_0$ are linearly independent. Thus, the linear independence of $a_0 - a_k,\ldots,a_{k-1} - a_k$ follows.

(ii) The barycentric coordinates of a vector x with respect to a pointwise independent set $A = \{a_0,\ldots,a_k\}$ are unique. For, suppose $x = \sum c_i a_i = \sum d_i a_i$, with $\sum c_i = \sum d_i = 1$. Then

$$0 = \sum(c_i - d_i)a_i = \sum(c_i - d_i)(a_i - a_0) + a_0\sum(c_i - d_i)$$
$$= \sum(c_i - d_i)(a_i - a_0)$$

By linear independence of $a_1 - a_0,\ldots,a_k - a_0$ we have $c_i - d_i = 0$ for $i = 1,\ldots,k$ and so $c_i = d_i$ for $i = 0,\ldots,k$.

<u>II.3. Definition</u>: Let $A = \{a_0,\ldots,a_k\}$ be a pointwise independent set in E^n. The set of all vectors in E^n whose barycentric coordinates with respect to A are non-negative (positive) is called the <u>Euclidean</u> <u>k-simplex</u> (open Euclidean k-simplex) <u>determined</u> <u>by</u> <u>A</u>, and is denoted by $[a_0 \ldots a_k]((a_0 \ldots a_k))$. The elements of

A are called the <u>vertices</u> of the simplex determined by A ; a point of $(a_0 \ldots a_k)$ is called an <u>inner point</u> of $[a_0 \ldots a_k]$.

<u>II.4. Remarks</u>: (i) With the <u>convex hull</u> of a set B in a linear space defined to be the set of all finite linear combinations $\sum t_i b_i$ of elements b_i of B , with real coefficients t_i satisfying $t_i \geq 0$ and $\sum t_i = 1$, a Euclidean k-simplex can be characterized as the convex hull of $k + 1$ pointwise independent vectors in E^n . It is then easy to see, for example, that a Euclidean 1-simplex is a closed line segment, a 2-simplex is a closed plane triangular region, and a 3-simplex is a solid tetrahedron. We shall furthermore adopt the convention of calling a single point a 0-simplex.

(ii) We assume the usual topology on E^n , and, unless otherwise indicated, a k-simplex s in E^n will be assumed to be topologized with the relative topology.

(iii) For a proof that the vertices of a Euclidean simplex are uniquely determined, see Bers [1], pp. 32-33.

II.5. Definition: (i) If $a_i, a_j, \ldots, a_m$ are $q + 1$ of the vertices of a Euclidean k-simplex $[a_0 \ldots a_k]$, the q-simplex $[a_i a_j \ldots a_m]$ is called a q-face of $[a_0 \ldots a_k]$. In particular, the $(k - 1)$-face of $[a_0 \ldots a_k]$ obtained by deleting a vertex a_i will be denoted by $[a_0 \ldots \hat{a}_i \ldots a_k]$.

(ii) A simplex s is a proper face of a simplex t if s is a face of t and $s \neq t$. This relationship will be written $s < t$.

II.6. Remark: It is clear from the uniqueness of barycentric coordinates that every point of a simplex is an inner point of a uniquely determined face of the simplex.

II.7. Definition: A finite collection K of Euclidean simplexes is said to be a Euclidean complex if the following two conditions are satisfied:

(i) the intersection of any two simplexes in K is either empty, or is a face of each of them;

(ii) if s is a simplex in K, then every face of s is also a member of K.

Examples of Euclidean complexes are:

(i) the collection of tetrahedrons,

triangles, edges, and vertices formed when a cube is partitioned by its diagonal planes; and

(ii) the collection of all proper faces of an $(n+1)$-simplex s^{n+1}. This latter complex will figure prominently in the sequel; we shall denote it by $B(s^{n+1})$.

<u>II.8. Definition</u>: A subset of a topological space is said to be an <u>n-cell</u>, an <u>open n-cell</u>, or an <u>(n-1)-sphere</u>, if it is homeomorphic, respectively, to the subsets $\{x = (x_1,\ldots,x_n) : \sum x_i^2 \leq 1\}$, $\{x : \sum x_i^2 < 1\}$, or $\{x : \sum x_i^2 = 1\}$, of E^n.

Thus, by the above definition, a closed interval is a 1-cell, a 0-sphere consists of two points, and a 1-sphere is a Jordan curve. By convention, a point is a 0-cell.

<u>II.9. Definition</u>: Let $K = \{t_i^p : p = 0,1,\ldots,a(K); i = 1,\ldots,b(p)\}$ be a finite collection of sets in a topological space. K is called a <u>topological complex</u> if the following properties are satisfied:

(i) t_i^p is a p-cell, for each p,i.

(ii) The intersection of any two

members of K is either empty or a member of K.

(iii) The boundary of each t_i^p, $p > 0$, is a $(p-1)$-sphere S_i^{p-1}, and is the union of sets t_j^q, $q = 0,1,\ldots,p-1$ of K, called <u>faces</u> of t_i^p. For each q, t_i^p has the same number of q-faces as does a Euclidean p-simplex. For $p = 0$, the boundary of t_i^p is the null set.

(iv) The spheres S_i^{p-1}, $p > 0$, are distinct. The members t_i^p of K are called <u>topological simplexes</u>.

<u>Example</u>: Partition the 2-sphere $S^2 = \{x : \sum x_i^2 = 1\}$ in E^3 by means of three mutually perpendicular planes. The collection K of curvilinear triangles, their edges, and vertices form a topological complex. The curvilinear triangles are topological 2-simplexes, the arcs which form their edges are topological 1-simplexes, and the vertices are topological 0-simplexes.

Note that, because of property (iv) in the definition, the partition of a 1-sphere by two points does not give a topological complex, while a partition by three points does.

II.10. Definition: (i) Let $\underline{K}$ denote the union of a collection K of simplexes (Euclidean or topological). If K is a complex, then $\underline{K}$ is called a *polyhedron*, and K is said to be a *triangulation* of, or *to triangulate*, $\underline{K}$. The same terminology applies to any homeomorph of $\underline{K}$.

(ii) The *dimension* of a complex K is the largest number n such that K contains an n-simplex.

Two important facts about complexes K are:

(i) $\underline{K}$ is compact and metrizable.

(ii) For each topological complex $K = \{t_i^p\}$ there is a Euclidean complex $L = \{s_i^p\}$ such that (a) there is a one-one correspondence between the simplexes t_i^p and s_i^p , and (b) there exists a homeomorphism h between $\underline{K}$ and $\underline{L}$ such that $h(t_i^p) = s_i^p$. (L is said to be a *Euclidean realization* of K .) For proofs of these facts, see Hocking and Young [1], Chapter 5, and Lefschetz [1], Chapter 3.

Because of (ii) above there is complete identification between topological and Euclidean complexes and simplexes, both topologically and in the algebraic structures to be introduced. Vertices

and simplexes may be identified, and barycentric coordinates introduced into topological complexes in an obvious way.

II.11. Definition: (i) Let K be a complex, and s a simplex of K. The collection of all simplexes in K having s as a face is called the *star of s* and is denoted by $*s$.

(ii) Let s be a simplex in K. The *star set* of s, denoted $\underline{*s}$ is the set of all points which are inner points of a simplex in $*s$.

(iii) If f is a continuous mapping of a polyhedron $\underline{K}$ into a polyhedron $\underline{L}$, then K is said to be *star-related to L relative to f* if, for each vertex p in K, there is a vertex v in L such that $f(\underline{*p})$ is contained in $\underline{*v}$.

(iv) Let K and L be complexes such that $\underline{K} = \underline{L}$. K is said to be a *refinement* of L, or to *refine* L, if, for every vertex p in K, there is a vertex v in L such that $\underline{*p}$ is contained in $\underline{*v}$.

II.12. Definition: A complex K is a *simplicial subdivision* of a complex L if:

(i) each simplex of K is contained in a

simplex of L ; and

(ii) $\underline{K} = \underline{L}$

We shall be concerned with a particular method of subdivision, known as <u>barycentric subdivision</u>, which we now describe. The process is described for Euclidean complexes, but in view of an earlier remark, the procedure carries over to topological complexes.

<u>II.13. Definition</u>: Let $s = [v_0 \ldots v_k]$ be a k-simplex. The <u>barycentre</u>, $\dot{s}$, is the point of s all of whose barycentric coordinates are equal; i.e., $\dot{s} = 1/(k+1)\sum v_i$.

<u>II.14. Definition</u>: Let $S(K) = \{\dot{s}_i\}$ be the set of barycentres of the simplexes s_i of a complex K .

(i) A subset $\{\dot{s}_j, \ldots, \dot{s}_k\}$ of $S(K)$ will be said to be an <u>admissible set</u> of $S(K)$ if it is a singleton, or the corresponding simplexes satisfy $s_j < \ldots < s_k$.

(ii) The convex hull of an admissible set $\{\dot{s}_i\}$ of $S(K)$, denoted $[\dot{s}_i]$, is called a <u>derived simplex</u> of K .

<u>II.15. Definition</u>: The collection K' of all derived simplexes of a complex K is called the <u>first barycentric</u>

subdivision of K .

II.16. Theorem: K' is a simplicial subdivision of K .

Proof: The proof is divided into three parts.

(1) A derived simplex is a simplex: Let $[\dot{s}_0 \ldots \dot{s}_k]$ be a derived simplex of K . Without loss of generality, we assume that s_i is an i-simplex and that the vertices v of s_k are so numbered that $s_i = [v_0 \ldots v_i]$, $i = 0,1,\ldots,k$. To show the pointwise independence of $\dot{s}_i, i = 0,\ldots,k$, we show that $\dot{s}_i - \dot{s}_0$, $i = 1,\ldots,k$, are linearly independent. If $a_i, i = 1, \ldots, k$, are real numbers such that $a_1(\dot{s}_1 - \dot{s}_0) + \ldots + a_k(\dot{s}_k - \dot{s}_0) = 0$, then

$$0 = \sum a_i(\dot{s}_i - \dot{s}_0) = \sum (a_i/(i+1))[(v_1 - v_0) + \ldots + (v_i - v_0)]$$
$$= \sum (a_i/(i+1) + \ldots + a_k/(k+1))(v_i - v_0)$$

Thus, from pointwise independence of $v_0,\ldots,v_k$, it follows that

$$a_i/(i+1) + \ldots + a_k/(k+1) = 0\,, \quad i = 1,\ldots,k\ .$$

Consequently $a_i = 0$ for $i = 1,\ldots,k$ and linear independence is shown.

(2) K' is a complex: That each face of a derived simplex is also a derived simplex, and hence in K' , follows

from the evident fact that a subset of an admissible set is also an admissible set. It remains to show that the intersection of derived simplexes is a derived simplex. This is accomplished by means of the following:

<u>Lemma</u>: Let s be an n-simplex.

(i) Every derived simplex whose vertices are barycentres of faces of s is contained in s .

(ii) Every point of s is an inner point of a unique derived simplex.

<u>Proof of Lemma</u>: (i) All barycentres of faces of s are contained in s ; since s is convex, s contains the convex hull of a set of such barycentres.

(ii) Let x be a point of s , and renumber the vertices v of s so that if

$$x = a_0 v_0 + \ldots + a_n v_n \quad , \text{ then } \quad 0 \leq a_0 \leq \ldots \leq a_n \quad .$$

Define inductively

$$b_0 = a_0 \quad , \qquad b_i = a_i - (b_0 + \ldots + b_{i-1}) \quad , \ i = 1,\ldots,n \ .$$

Then, for $i = 1,\ldots,n$, $b_i = a_i - a_{i-1} \geq 0$, $b_i > 0$ if and only if $a_i > a_{i-1}$, and $a_i = b_0 + \ldots + b_i$.

Thus,

$$\begin{aligned} x &= b_0 v_0 + (b_0 + b_1) v_1 + \ldots + (b_0 + \ldots + b_n) v_n \\ &= b_0 (v_0 + \ldots + v_n) + \ldots + b_i (v_i + \ldots + v_n) \\ &\qquad + \ldots + b_n v_n \\ &= (n+1) b_0 \dot{s}_0 + \ldots + (n-i+1) b_i \dot{s}_i + \ldots + b_n \dot{s}_n \end{aligned}$$

where s_i is the $(n-i)$-simplex $[v_i \dots v_n]$, $i = 0,\dots,n$. Since $(n + 1)b_0 + nb_1 + \dots + b_n = a_0 + \dots + a_n = 1$, it follows that x is in the derived simplex $[\dot{s}_n \dots \dot{s}_0]$. If t is the derived simplex with vertices $\dot{s}_i$, for those i for which $b_i > 0$, then x is clearly an inner point of t . Uniqueness of t follows by reversing the above computation.

We now return to the proof of (2). Suppose first that t,u are derived simplexes contained in the same simplex s of K . If there is a point of $t \cap u$ which is an inner point of both t and u , then by the Lemma, $t = u$; if there is a point of $t \cap u$ which is an inner point of t but not of u , then from Remark II.6 and the Lemma, it follows that t is a face of u . In either case $t \cap u = t$ is a derived simplex. In the case remaining, where no point of $t \cap u$ is an inner point of t or of u , it follows that $t \cap u$ is contained in a proper p-face t^* of t and a proper q-face u^* of u , where p,q are chosen to be minimal. Applying the above reasoning to the derived simplexes t^* and u^* , it follows that $t^* = u^*$, t^* is a face of u^* , or there is a proper p^*-face t^{**} of t^* and a proper q^*-face u^{**} of u^* such that $t \cap u = t^* \cap u^* \subset t^{**} \cap u^{**}$. The last case contradicts the minimality of p and q and is hence rejected. In either of the other cases, $t \cap u = t^* \cap u^* = t^*$ is a derived simplex.

Now, suppose t, u are contained in the simplexes s, s', respectively, of K. Note that the intersection of t with a face s_1 of s is a face of t and hence a derived simplex, with a similar statement for u and a face s'_1 of s'. (Proof: Let T be the set of vertices of t, B the set of barycentres of faces of s_1, and $D = T \cap B$. From convexity it is clear that $t \cap s_1$ contains $[D]$. Conversely, let x be an element of $t \cap s_1$. By part (ii) of the lemma, x is an inner point of a unique simplex $[B_1]$, where B_1 is some subset of B, and by Remark II.6, x is also an inner point of a unique face $[T_1]$ of t, where T_1 is a subset of T. Since $[B_1]$ and $[T_1]$ are derived simplexes of s, it follows, again from the lemma, that $[T_1] = [B_1]$. The vertices of a simplex are uniquely determined, however, so we must have $T_1 = B_1$. It follows immediately that x is an element of $[D]$.)

Consequently, since $s \cap s' = r$ is a simplex of K,

and $$t \cap u = (t \cap r) \cap (u \cap r) ,$$

$t \cap u$ is the intersection of two derived simplexes contained in the same simplex of K, and we have the first case considered above.

(3) K' subdivides K simplicially: It is an immediate result of the lemma proved in (2) that the conditions of Definition II.12 are satisfied.

The process of barycentric subdivision may be applied in turn to the complex K' to obtain the second barycentric subdivision K'', to K'' to obtain the third barycentric subdivision, and so on. We denote the <u>k-th barycentric subdivision</u> of K by $K^{(k)}$.

<u>II.17. Definition</u>: (i) The <u>diameter</u> of a set A in a metric space (X,d) is defined by

$$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\} .$$

(ii) The <u>mesh</u> of a complex K is defined to be the maximum of the diameters of the simplexes in K.

<u>II.18. Lemma</u>: The diameter of a simplex is the length of its longest edge (i.e., 1-face).

<u>Proof</u>: Let $s = [v_0 \ldots v_n]$ be a simplex and let $d = |v_i - v_j|$ be the length of the longest 1-face $[v_i v_j]$. It suffices to show that $|x' - y'| \leq d$ for any x', y' in s.

Let v_p be any vertex of s; it is clear that $D(v_p, d) = \{z : |v_p - z| \leq d\}$ is a convex set which contains $v_m, m = 0, \ldots, n$, and hence contains

s . Thus, $|v_p - x'| \leq d$, which implies conversely that v_p is in $D(x,d)$. Therefore, $D(x',d)$ is a convex set which contains $v_m, m = 0,\ldots,n$ and hence also contains s . Since y' is in s , we have $|x' - y'| \leq d$, which was to be shown.

The importance of the barycentric method of subdivision arises from the following theorem.

II.19. Theorem: mesh $K^{(n)} \to 0$ as $n \to \infty$.

Proof: It clearly suffices to consider a complex K consisting of an k-simplex s together with all its faces.

We first show that

$$\text{mesh } K' \leq kd/(k+1)$$

where d is the diameter of s .

Let $\dot{s}_i, \dot{s}_j$ be vertices of a derived simplex of K .

We suppose the vertices of s renamed so that

$s_i = [v_0 \ldots v_i]$, $s_j = [v_0 \ldots v_j]$ with $i < j \leq k$

Then, using Lemma II.18, we have

$$|\dot{s}_i - \dot{s}_j| = |1/(i+1) \sum_{m=0}^{i} v_m - \dot{s}_j|$$

$$= 1/(i+1) |\sum (v_m - \dot{s}_j)|$$

$$= 1/(i+1)(j+1) |\sum_{m=0}^{i} \sum_{p=0}^{j} (v_m - v_p)|$$

$$\leq 1/(i+1)(j+1) \sum \sum |v_m - v_p|$$

$$\leq [(i+1)(j+1)-(i+1)]d/(i+1)(j+1)$$

$$= jd/(j+1) \leq kd/(k+1)$$

Thus, mesh $K' \leq kd/(k+1)$; repeated application gives

$$\text{mesh } K^{(n)} \leq [k/(k+1)]^n d \quad ,$$

and the theorem follows.

We are now ready to associate an algebraic structure with a complex; to do this, we begin with the definitions of an oriented complex and incidence numbers.

<u>II.20. Definition</u>: (i) Let $s = [a_0 \ldots a_p]$ be a simplex, and choose an arbitrary fixed ordering for the vertices a_i ; s is then said to be <u>oriented</u>. The class of all orderings of the vertices obtained through an even permutation of the fixed ordering will be called the <u>positively oriented simplex s</u> and is denoted by $+s$. The class of orderings obtained through odd permutations of the fixed ordering is called the <u>negatively oriented simplex s</u> and is denoted by $-s$.

(ii) An <u>oriented complex</u> is a complex in which each simplex has been assigned an

arbitrary, fixed orientation.

II.21. Remark: A complex may be oriented by assigning an orientation to the individual simplexes without regard to how the simplexes are joined, or whether one simplex is a face of another.

A complex can also be oriented by giving its vertices a fixed order, and the positive orientation of the simplexes then taken to be the naturally induced ordering.

II.22. Definition: (i) Let K be an oriented complex, and let $s = [a_0 \ldots a_m]$ be an m-simplex in K and $t = [v_0 \ldots v_{m-1}]$ an (m-1)-simplex of K. The incidence number, (s,t), associated with s and t is defined as follows:

$(s,t) = 0$ if t is not a face of s

$(s,t) = \pm 1$ if $t = [a_0 \ldots \hat{a}_i \ldots a_m]$

and $[a_i a_0 \ldots \hat{a}_i \ldots a_m] = \pm s$.

(ii) t is called a positively oriented or negatively oriented face of s, depending on whether $(s,t) = 1$ or $(s,t) = -1$.

Example: Let K be a 3-simplex $s = [a_0 a_1 a_2 a_3]$

together with its sides and vertices. If the order $a_0a_1a_2a_3$ is assigned to the vertices, this induces the following orderings on the sides:

$$a_0a_1a_2 \quad \text{on} \quad s_1 = [a_0a_1a_2] \quad ,$$

$$a_0a_1a_3 \quad \text{on} \quad s_2 = [a_0a_1a_3] \quad ,$$

$$a_0a_2a_3 \quad \text{on} \quad s_3 = [a_0a_2a_3] \quad ,$$

$$a_1a_2a_3 \quad \text{on} \quad s_4 = [a_1a_2a_3] \quad .$$

Computing the incidence numbers, we have

$$[a_3a_0a_1a_2] = -[a_0a_1a_2a_3]$$

since the ordering $a_3a_0a_1a_2$ is obtained from $a_0a_1a_2a_3$ by the odd permutation $(0,3,2,1)$.

Similarly, $a_2a_0a_1a_3$ is obtained by the even permutation $(0,2,1)$ so that $[a_2a_0a_1a_3] = +s$; $a_1a_0a_2a_3$ is obtained by the odd permutation $(0,1)$ so that $[a_1a_0a_2a_3] = -s$; and $[a_0a_1a_2a_3] = +s$.

Thus $(s,s_1) = -1 = (s,s_3)$

$$(s,s_2) = 1 = (s,s_4) \quad .$$

<u>II.23. Lemma</u>: Let $s_i = [a_0 \ldots \hat{a}_i \ldots a_n]$ be an $(n-1)$-face of an n-simplex $s = [a_0 \ldots a_n]$. Then $(s,s_i) = (-1)^i$.

Proof: If $i = 0$, the result is obvious; assume $i = 1,\ldots,n$. The ordering $a_i a_0 \ldots \hat{a}_i \ldots a_n$ is obtained from the ordering $a_0 \ldots a_n$ by the $(i+1)$-cycle $p = (a_0, a_i, a_{i-1}, \ldots, a_1)$. If i is odd, $i + 1$ is even and p is an odd permutation; conversely, if i is even $i + 1$ is odd and p is an even permutation. Thus, $[a_i a_0 \ldots \hat{a}_i \ldots a_n] = (-1)^i s$, so by definition $(s, s_i) = (-1)^i$.

II.24. Definition: Let K be an oriented complex and let G be an additively written abelian group.

(i) An m-dimensional chain on K with coefficients in G is a function c_m defined on the m-simplexes of K, with values in G, such that if $c_m(s^m) = g$, then $c_m(-s^m) = -g$.

(ii) An elementary m-chain on K is an m-chain c_m such that for some m-simplex s_0^m of K,

$$c_m(\pm s_0^m) = \pm g_0 , \text{ and } c_m(s^m) = 0 \text{ for } s^m \neq s_0^m .$$

II.25. Remarks: (i) With addition of m-chains taken to be the usual pointwise addition of functions,

$$(c_m^1 + c_m^2)s^m = c_m^1(s^m) + c_m^2(s^m) ,$$

it is easy to verify that the collection of m-chains

on a complex K, with coefficients in a given group G, forms an abelian group. This group is called the <u>m-dimensional chain group of K with coefficients in G</u>, and is denoted by $C_m(K,G)$. If K has no m-simplexes, we take $C_m(K,G)$ to be the trivial group $\{0\}$.

(ii) Let c_m be an elementary m-chain on a complex K, with $c_m(s_0^m) = g_0$, $c_m(s^m) = 0$ for $s^m \neq s_0^m$. We shall denote such an m-chain by the formal product $g_0 s_0^m$; an arbitrary m-chain c_m will then be written as a linear combination $\sum g_i s_i^m$ where $g_i = c_m(+ s_i^m)$. We shall also regard a p-simplex s of K as an elementary p-chain, with values $+1$ on s, 0 elsewhere.

<u>II.26. Definition</u>: The <u>boundary operator</u> $b_m = b$ is a function defined on the elementary m-chains by

$$b(g_0 s_0^0) = 0 \quad ; \quad b(g_0 s_0^m) = \sum (s_0^m, s^{m-1}) g_0 s^{m-1}, m > 0 ,$$

where (s_0^m, s^{m-1}) is the incidence number and the summation is taken over all (m-1)-faces of s^m. The definition of b is then extended linearly to arbitrary m-chains:

$$b(\sum g_i s_i^m) = \sum b(g_i s_i^m) .$$

<u>II.27. Theorem</u>: For any m-chain $c_m, b(b(c_m)) = 0$.

<u>Proof</u>: Since b is linear, it suffices to prove the theorem for an arbitrary elementary m-chain gs^m , where $s^m = [v_0 \ldots v_m]$. From Lemma II.23, it is clear that the defining relation for b can be written

$$b(gs^m) = \sum_i g(-1)^i [v_0 \ldots \hat{v}_i \ldots v_m] \quad .$$

Thus we have

$$\begin{aligned} b(b(gs^m)) &= \sum_i g(-1)^i b([v_0 \ldots \hat{v}_i \ldots v_m]) \\ &= \sum_i g(-1)^i \{ \sum_{j<i} (-1)^j [v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_m] \\ &\qquad + \sum_{i<j} (-1)^{j-1} [v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_m] \} \end{aligned}$$

The signs $(-1)^j$ for $j<i$ and $(-1)^{j-1}$ for $i<j$ occur according to the lemma, since in the first case the deleted vertex v_j is still in the $j+1$ position in the simplex; while in the second case, since v_i has already been deleted, v_j has been moved down to the j position.

Furthermore, each (m-2)-simplex $[v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_m]$ appears twice, once with coefficient $g(-1)^{i+j}$ and once with coefficient $g(-1)^{i+j-1}$. Consequently, $b(b(gs^m)) = 0$.

<u>II.28. Theorem</u>: b is a homomorphism of $C_m(K,G)$

into $C_{m-1}(K,G)$ (where $C_{m-1}(K,G) = \{0\}$ if $m = 0$) .

Proof: Follows trivially from the definition of b .

II.29. Definition: Since b is a homomorphism, the kernel of b (i.e., the set of all c_m in $C_m(K,G)$ such that $b(c_m) = 0$), is an abelian subgroup of $C_m(K,G)$, called the m-dimensional cycle group of K with coefficients in G , and is denoted by $Z_m(K,G)$. An element of $Z_m(K,G)$ is called an m-dimensional cycle, or m-cycle, on K .

II.30. Definition: An m-chain c_m is called an m-boundary if there is an (m+1)-chain c_{m+1} such that $c_m = b(c_{m+1})$. If $B_m(K,G)$ denotes the set of m-boundaries on K , with coefficients in G , then $B_m(K,G)$ is the image of $C_{m+1}(K,G)$ under b , and is hence a subgroup of $c_m(K,G)$, called the group of m-boundaries on K .

II.31. Remarks: If c_m is an m-boundary on K , then for some c_{m+1} in $C_{m+1}(K,G)$, $c_m = b(c_{m+1})$, hence $b(c_m) = b(b(c_{m+1})) = 0$ by Theorem II.27. Thus $B_m(K,G)$ is a subgroup of $Z_m(K,G)$.

As subgroups of the abelian group $C_m(K,G)$, both $Z_m(K,G)$ and $B_m(K,G)$ are abelian, so we can form the factor group $H_m(K,G) = Z_m(K,G)/B_m(K,G)$.

<u>II.32. Definition</u>: Let K be a complex and G an abelian group.

(i) The factor group $H_m(K,G)$ is called the <u>m-th homology group of K over G</u>.

(ii) Two cycles z,z' in $Z_m(K,G)$ are said to be <u>homologous</u> if $z-z'$ is in $B_m(K,G)$.

(iii) If G is taken to be the group of integers, the groups $C_m(K,G),\ldots,H_m(K,G)$ are called the <u>integral m-chain group</u>,..., the <u>integral m-homology group</u>, and are written $C_m(K),\ldots,H_m(K)$.

<u>II.33. Definition</u>: Let $a_0,\ldots,a_n$ be vertices of a complex K. The convex hull $s = [a_0\ldots a_n]$ is called a <u>degenerate n-simplex</u> if the a_i, $i = 0,\ldots,n$, are not pointwise independent. By convention, we set gs equal to the zero n-chain if s is degenerate, and define $b([a_0\ldots a_n])$ to be $\sum(-1)^i[a_0\ldots\hat{a}_i\ldots a_n]$.

<u>II.34. Definition</u>: (i) Let $s = [a_0\ldots a_n]$ be an oriented simplex in a complex K, and let v be a vertex of K. If $[va_0\ldots a_n]$ is an oriented simplex

of K, then it is denoted by vs and called the <u>cone over s with vertex v</u>. If $[va_0 \ldots a_n]$ is degenerate, vs is said to be a <u>degenerate</u> cone over s.

(ii) Let $c_n = \sum g_i s_i$ be an n-chain on K and let v be a vertex of K. Then the <u>cone over c_n with vertex v</u>, denoted vc_n, is given by

$$vc_n = \sum g_i vs_i \quad .$$

vc_n is <u>degenerate</u> if some vs_i is degenerate. We define $b(vc_n)$ to be $\sum g_i b(vs_i)$, whether vc_n is degenerate or not.

<u>II.35. Lemma</u>: $b(vc_n) = c_n - vb(c_n)$.

<u>Proof</u>: By linearity of b it suffices to assume c_n is an elementary n-chain, $c_n = gs$, with $s = [a_0 \ldots a_n]$. Then we have

$$\begin{aligned} b(vgs) = b(gvs) &= gb(vs) \\ &= gb([va_0 \ldots a_n]) \\ &= g([a_0 \ldots a_n] + \sum(-1)^{i+1}[va_0 \ldots \hat{a}_i \ldots a_n]) \\ &= g(s - \sum(-1)^i v[a_0 \ldots \hat{a}_i \ldots a_n]) \\ &= g(s - vb(s)) = c_n - vb(c_n) \quad . \end{aligned}$$

II.36. Example: Let K be the complex $B(s^{n+1})$ with $s^{n+1} = [a_0 \ldots a_{n+1}]$ (see example (ii) following Definition II.7). The integral homology groups $H_p(K)$ are infinite cyclic for $p = 0, n$ and are $\{0\}$ for $p = 1, \ldots, n-1$.

Proof: (i) $H_0(K)$ is cyclic: Note that every 0-chain is a 0-cycle, i.e., $C_0(K) = Z_0(K)$. Thus if $c = \sum g_i a_i$ is a 0-chain, it suffices to show that for $g = \sum g_i$, c is homologous to $c_0 = ga_0$; for, if so, then $c - c_0$ is in $B_0(K)$ and we have

$$c + B_0(K) = c_0 + B\ (K) = ga_0 + B_0(K) = g(a_0 + B_0(K)) \quad .$$

We have

$$c - c_0 = \sum_{i=1}^{n+1} g_i(a_i - a_0) = \sum g_i b([a_0 a_i])$$

$$= b(\sum g_i [a_0 a_i]) \; ,$$

so that $c - c_0$ is the boundary of a 1-chain on K , and c is homologous to c_0 .

(ii) $H_p(K) = \{0\}$, $0 < p < n$: Let z be a cycle on K , and let v be a vertex of K . Now, z is a linear combination of simplexes of K , some of which may have v as a vertex.

We may thus write, for suitable chains c_p, c_{p-1}

$$z = vc_{p-1} + c_p$$

where v is not a vertex of any simplex in c_{p-1} or

c_p . Consequently, since z is a cycle, and by Lemma II.35,

$$0 = b(z) = c_{p-1} - vb(c_{p-1}) + b(c_p) \quad .$$

Now, $vb(c_{p-1})$ is a linear combination of simplexes with vertex v, while no simplex of c_{p-1} or $b(c_p)$ has v as a vertex, so we must have $vb(c_{p-1}) = 0$, $c_{p-1} + b(c_p) = 0$.

Therefore $z = vc_{p-1} + c_p = -vb(c_p) + c_p = b(vc_p)$.

Hence z is a p-boundary, and we have $z_p(K) = B_p(K)$.

Thus $H_p(K) = Z_p/Z_p = \{0\}$.

(iii) $\underline{H_n(K)}$ <u>is infinite cyclic</u>: Since $B_n(K) = \{0\}$ (there are no (n+1)-simplexes in K, so $C_{n+1}(K) = \{0\}$), it follows that $H_n(K) = Z_n(K)$. Thus we have only to show $Z_n(K)$ is cyclic. Let z be the n-chain $\sum(-1)^i[a_0 \ldots \hat{a}_i \ldots a_{n+1}]$. Then by the same computation as in the proof of Theorem II.27, we have $b(z) = 0$; i.e., z is an n-cycle. (Note that if K^* is the complex $s^{n+1} \cup K$, then in K^*, $z = b(s^{n+1})$) . It is clear that every n-chain c can be written in the form

$$c = \sum_{i=0}^{n+1} g_i(-1)^i[a_0 \ldots \hat{a}_i \ldots a_{n+1}] \quad .$$

If c is a cycle, we have

$$0 = b(c) = \sum_{i=0}^{n+1} g_i(-1)^i \sum_j (-1)^k [a_0 \ldots \hat{a}_j \ldots \hat{a}_i \ldots a_{n+1}]$$

where the summation on j is taken over the range $0,\ldots,\hat{\imath},\ldots,n+1$, and $k = j$ if $j < i$, $k = j-1$ if $i < j$.

Denoting $[a_0 \ldots \hat{a}_j \ldots \hat{a}_i \ldots a_{n+1}]$ by s_{ij}, we have

$$0 = b(c) = \sum_i \sum_j g_i (-1)^{k+i} s_{ij} .$$

Since $s_{ij} = s_{ji}$, s_{ij} appears twice, once with coefficient $g_i(-1)^{i+j-1}$, $(i < j)$, and again with coefficient $g_j(-1)^{i+j}$, $(j < i)$. Therefore each s_{ij} has coefficient $\pm (g_i - g_j)$, which is necessarily equal to zero. Since this is true for all possible combinations of i and j, we have $g_0 = \ldots = g_{n+1}$.

Denoting the common value by g, we have $c = gz$, and $Z_n(K)$ is infinite cyclic. Note that if w is an n-cycle with the property that for each z_n in $Z_n(K)$ there is a g_n such that $z_n = g_n w$, then we have

$$z = gw , \quad w = g'z , \text{ for integers } g , g' .$$

Hence $z = gg'z$ implies $1 - gg' = 0$, and $g = g' = \pm 1$. Consequently there are only two generating n-cycles, z and $-z$. We shall refer to $z = \sum(-1)^i[a_0 \ldots \hat{a}_i \ldots a_{n+1}]$ as <u>the fundamental cycle</u>

on K .

II.37. Definition: Let $\underline{K}$ and $\underline{L}$ be polyhedra, with triangulations K and L respectively. A continuous mapping f from $\underline{K}$ into $\underline{L}$ is a simplicial mapping if conditions (i) and (ii) are satisfied:

(i) for each simplex $[a_0 \ldots a_p]$ in K, $f(a_0), \ldots, f(a_p)$ are the vertices of a simplex in L. (If $f(a_i) = f(a_j)$, $i \neq j$, then $[f(a_0) \ldots f(a_p)]$ is degenerate, and f is said to collapse $[a_0 \ldots a_p]$) ;

(ii) for $x = \sum c_i a_i$, where c_i are the barycentric coordinates of x with respect to the vertices of a simplex $[a_0 \ldots a_p]$ in K, $f(x) = \sum c_i f(a_i)$.

II.38. Definition: Let K,L be complexes, G an abelian group, and for each p let F_p be a homomorphism of $C_p(K,G)$ into $C_p(L,G)$. The collection $F = \{F_p\}$ is called a chain mapping of K into L if for each c_p in $C_p(K,G)$,

$$b(F_p(c_p)) = F_{p-1}(b(c_p)) \quad , \quad p = 0,1,2,\ldots,$$

where $F_{-1} = 0$. We shall henceforth adopt the convention of writing $F(c_p)$ for $F_p(c_p)$ and speaking of the "homomorphism" F .

<u>II.39. Theorem</u>: A simplicial mapping f of $\underline{K}$ into $\underline{L}$ induces a chain mapping F of $C_p(K,G)$ into $C_p(L,G)$.

<u>Proof</u>: If $s = [a_0 \ldots a_p]$ is a simplex in K , define for the elementary p-chain gs ,

$$F_p(gs) = g[f(a_0) \ldots f(a_p)]$$

which is an elementary p-chain of $C_p(L,G)$ (the zero elementary p-chain if f collapses s). Then for any p-chain $\sum g_i s_i^p$, extend the definition of F_p linearly: $F_p(\sum g_i s_i^p) = \sum F_p(g_i s_i^p)$. It is clear from the linear definition that each F_p is a homomorphism. We now show that for $F \varepsilon \{F_p\}$, $bF = Fb$. Let c_p be a p-chain on K ; it suffices to assume $c_p = gs$, $s = [a_0 \ldots a_p]$.

Suppose first that f does not collapse s . By definition we have

$$\begin{aligned} b(F(gs)) &= b(g[f(a_0) \ldots f(a_p)] \\ &= \sum (-1)^i g[f(a_0) \ldots (f(a_i))\hat{}\ldots f(a_p)] \quad , \end{aligned}$$

while

$$\begin{aligned} F(b(gs)) &= F(\sum (-1)^i g[a_0 \ldots \hat{a}_i \ldots a_p]) \\ &= \sum (-1)^i F(g[a_0 \ldots \hat{a}_i \ldots a_p]) \\ &= \sum (-1)^i g[f(a_0) \ldots (f(a_i))\hat{}\ldots f(a_p)] \quad . \end{aligned}$$

If f collapses s, then $f(a_i) = f(a_j)$, for some $i,j,i \neq j$, and hence

$$b(F(gs)) = b(0) = 0 \quad .$$

We assume without loss of generality that $f(a_0) = f(a_1)$, since the orientation of $[a_0 \ldots a_p]$ is equivalent to the orientation of one of $[a_i a_j a_0 \ldots \hat{a}_i \ldots \hat{a}_j \ldots a_p]$ or $[a_j a_i a_0 \ldots \hat{a}_i \ldots \hat{a}_j \ldots a_p]$. Then

$$\begin{aligned}
F(b(gs)) &= F(\sum_{i=0}^{p}(-1)^i g[a_0 \ldots \hat{a}_i \ldots a_p]) \\
&= F(g[a_1 a_2 \ldots a_p]) - F(g[a_0 a_2 \ldots a_p]) \\
&\qquad + \sum_{i=2}^{p}(-1)^i F(g[a_0 a_1 \ldots \hat{a}_i \ldots a_p]) \\
&= g[f(a_1) \ldots f(a_p)] - g[f(a_0) f(a_2) \ldots f(a_p)] \\
&\qquad + \sum_{i=2}^{p}(-1)^i g[f(a_0) f(a_1) \ldots (f(a_i))\hat{} \ldots f(a_p)] \\
&= 0 \quad ,
\end{aligned}$$

since the first two terms cancel and each simplex in the summation is degenerate.

II.40. Lemma: Let F be a chain mapping of K into L. Then F preserves cycles, boundaries, and the homology relation between cycles.

Proof: Let z be a cycle in $Z_p(K,G)$; since F commutes with the boundary operator, we have

$$b(F(z)) = F(b(z)) = F(0) = 0$$

and $F(z)$ is in $Z_p(L,G)$.

If c_p is a p-boundary, then for some $(p+1)$-chain c_{p+1} , $c_p = b(c_{p+1})$. Thus

$$F(c_p) = F(b(c_{p+1})) = b(F(c_{p+1})) ,$$

and since $F(c_{p+1})$ is in $C_{p+1}(L,G)$, it follows that $F(c_p)$ is in $B_p(L,G)$.

Now suppose $z_1 - z_2$ is in $B_p(K,G)$, i.e., z_1, z_2 are homologous cycles. Then $z_1 - z_2 = b(c_{p+1})$ for some c_{p+1} in $C_{p+1}(K,G)$. Hence,

$$\begin{aligned} F(z_1) - F(z_2) = F(z_1 - z_2) &= F(b(c_{p+1})) \\ &= b(F(c_{p+1})) , \end{aligned}$$

so that $F(z_1) - F(z_2)$ is in $B_p(L,G)$.

<u>II.41. Theorem</u>: A chain mapping F of K into L induces a homomorphism F^* from $H_p(K,G)$ into $H_p(L,G)$.

<u>Proof</u>: Any element h of $H_p(K,G)$ can be represented in the form $h = z + B_p(K,G)$, for some p-cycle z . Define

$$F^*(h) = F^*(z + B_p(K,G) = F(z) + B_p(L,G) .$$

It is clear that F^* is a homomorphism, since F is. We show F^* to be well-defined.

Suppose $h = z + B_p(K,G) = z' + B_p(K,G)$. Then $z - z'$ is in $B_p(K,G)$ so that, by the preceding lemma, $F(z) - F(z')$ is in $B_p(L,G)$. This means that $F(z) + B_p(L,G) = F(z') + B_p(L,G)$, which was to be shown.

It is clear from the definitions that if f,g are chain mappings:

$$f : C_p(K,G) \to C_p(L,G) \ ; \ g : C_p(L,G) \to C_p(M,G) \quad ,$$

then the composite mapping

$$gf : C_p(K,G) \to C_p(M,G)$$

is a chain mapping, and

$$g^*f^* = (gf)^* \quad .$$

<u>II.42. Definition</u>: Let K be a complex, K' its first barycentric subdivision. The <u>chain</u> <u>derivation</u>, or <u>chain</u> <u>subdivision</u> mapping d from $C_p(K,G)$ into $C_p(K',G)$ is defined recursively as follows:

(i) for a vertex v of K , $d(v) = \dot{v} = v$

(ii) For an elementary p-chain gs^p , $p > 0$

$$d(gs^p) = g\dot{s}^p d(bs^p)$$

(Note that $d(gs^p)$ is a p-chain of K' consisting of terms $\pm\, gs_i^p$, where s_i^p ranges over the derived p-simplexes contained in s^p , and the sign is determined by orientation. Consequently $d(gs^p)$ is not degenerate.)

(iii) The definition of d is extended linearly to arbitrary p-chains.

The chain derivation mapping d may be applied in turn to the chains on K', then to the chains on $K^{(2)}$, and so on. If $K^{(m)}$ is any barycentric subdivision of K, the m-th iterate of d will also be denoted by d and referred to as the <u>chain derivation</u> mapping from $C_p(K,G)$ to $C_p(K^{(m)},G)$.

<u>II.43. Theorem</u>: The chain derivation mapping d is a chain mapping from K to K'.

<u>Proof</u>: That d is a homomorphism is clear from the definition. It remains to show that $db = bd$. To this end, let s^p be an elementary p-chain; if $p = 0$ it is clear that

$$d(b(s^p)) = d(0) = 0 = b(s^p) = b(d(s^p)) \quad .$$

Assuming that $d(b(s^q)) = b(d(s^q))$ for all $q < p$, we have by Lemma II.35,

$$b(d(s^p)) = b(\dot{s}^p d(b(s^p))) = d(b(s^p)) - \dot{s}^p b(d(b(s^p))) \quad .$$

By the induction hypothesis,

$$b(d(b(s^p))) = d(b(b(s^p))) = d(0) = 0 \quad ,$$

so that $b(d(s^p)) = d(b(s^p))$, and the commutivity of b and d follows.

<u>Example</u>: We compute $d(s)$, where s is the 2-simplex $[a_0a_1a_2]$, and s is oriented by the ordering $a_0a_1a_2$.

$$d(s) = \dot{s}d(b[a_0a_1a_2])$$
$$= \dot{s}d([a_1a_2] - [a_0a_2] + [a_0a_1])$$

Denoting the barycentre of $[a_ia_j]$ by a_{ij}, we have

$$d[a_ia_j] = a_{ij}d(b[a_ia_j]) = a_{ij}d(a_j - a_i)$$
$$= [a_{ij}a_j] - [a_{ij}a_i]$$

Thus,

$$d(s) = [\dot{s}a_{12}a_2] - [\dot{s}a_{12}a_1] - [\dot{s}a_{02}a_2] + [\dot{s}a_{02}a_0] + [\dot{s}a_{01}a_1] - [\dot{s}a_{01}a_0]$$

<u>II.44. Theorem</u>: Let f be a function from <u>K</u>' to <u>K</u> such that for each vertex $\dot{s}$ in K', $f(\dot{s})$ is a vertex of s, and, for $x = \sum c_i\dot{s}_i$, $f(x) = \sum c_i f(\dot{s}_i)$, where $\dot{s}_i$ are the vertices of the derived simplex containing x.

Then

(i) f is a simplicial mapping from K' onto K;

(ii) the chain mapping d' induced by f satisfies $d'd = I$, where I is the identity mapping.

<u>Proof</u>: (i) If $[\dot{s}_m\ldots\dot{s}_k]$ is a derived simplex then $s_m < \ldots < s_k$, so that $f(\dot{s}_m)\ldots f(\dot{s}_k)$ are vertices of s_k.

(ii) It suffices to show that for an elementary p-chain s, $d'(d(s)) = s$. If $p = 0$, this is obvious. Proceeding by induction, assume the equality holds for $(p-1)$-chains. Now, each vertex of a derived simplex t_i in $d(s)$ is a barycentre of s or a face of s; therefore $d'(t_i)$ is $\pm s$ or zero. Consequently, for some integer n, $d'(d(s)) = ns$. By the induction hypothesis, we have

$$nb(s) = b(ns) = b(d'(d(s))) = d'(d(b(s))) = b(s).$$

Thus $n = 1$, and we have $d'(d(s)) = s$.

<u>II.45. Definition</u>: A mapping d' defined as in Theorem II.44 is called <u>an inverse</u> of the chain derivation mapping d.

<u>II.46. Definition</u>: Let f be a continuous mapping from a polyhedron $\underline{K}$ to a polyhedron $\underline{L}$. A simplicial mapping $g : \underline{K} \to \underline{L}$ is said to be a <u>simplicial approximation to f</u> relative to K and L if, for every x in $\underline{K}$, $f(x)$ and $g(x)$ belong to the same simplex of L.

<u>II.47. Remark</u>: Recall that two mappings f,g of a topological space X into a topological space Y are said to be <u>homotopic</u> if there is a continuous mapping h from $X \times [0,1]$ into Y such that

$h(x,0) = f(x)$, $h(x,1) = g(x)$, for all x in X .

Let f,g,K,L be as in the preceding definition, and for x in $\underline{K}$, let $s(x)$ be the simplex of L which contains $f(x)$ and $g(x)$. By convexity of $s(x)$, $f(x)$ and $g(x)$ are endpoints of a segment contained in $s(x)$ and hence in $\underline{L}$.

Thus the mapping

$$h(x,t) = tf(x) + (1-t)g(x)$$

is a homotopy from $\underline{K} \times [0,1]$ to $\underline{L}$, and we conclude that a simplicial approximation g to a continuous mapping f between polyhedra is homotopic to f .

<u>II.48. Lemma</u>: Let $a_0,\ldots,a_r$ be 0-simplexes in a complex K . Then the a_i are the vertices of a simplex in K if and only if the star sets $\underline{*a_i}$ have nonempty intersection.

<u>Proof</u>: Suppose the a_i are the vertices of a simplex s in K . Then s belongs to $*a_i$ for each $i = 0,\ldots,r$, so that if x is an inner point of s , then x is in $\underline{*a_i}$ for each i , and $\cap\ \underline{*a_i}$ is nonempty.

Conversely, let $x \varepsilon \cap \underline{*a_i}$. Then $x \varepsilon \underline{*a_i}$,

$i = 0,\ldots,r$ and therefore belongs to a simplex s of K which has $a_0 \ldots a_r$ among its vertices. Consequently $[a_0 \ldots a_r]$ is a face of s and belongs to K.

<u>II.49. Theorem</u>: Suppose f is a continuous mapping of $\underline{K}$ into $\underline{L}$ and K is star-related to L relative to f. Then there is a simplicial mapping g from $\underline{K}$ to $\underline{L}$ which is a simplicial approximation to f.

<u>Proof</u>: For any vertex p of K there is a vertex $v = v(p)$ of L such that $f(\underline{*p})$ is contained in $\underline{*v}$. Choose one such v for each p and define

$$g'(p) = v \; .$$

If $[p_0 \ldots p_r]$ is a simplex in K, then $\cap \underline{*p_i}$ is nonempty, and we have

$$f(\cap \underline{*p_i}) \subset \cap f(\underline{*p_i}) \subset \cap *[\underline{g'(p_i)}]$$

so that $g'(p_i)$ are the vertices of a simplex in L. Extend g' linearly to obtain a simplicial mapping g of $\underline{K}$ into $\underline{L}$.

Now, let $x = a_0 p_0 + \ldots + a_r p_r$ be an inner point of some simplex $[p_0 \ldots p_r]$ of K. Then $x \varepsilon \cap \underline{*p_i}$ and we have

$$f(x) \varepsilon f(\cap \underline{*p_i}) \subset \cap f(\underline{*p_i}) \subset \cap \underline{*[g(p_i)]} \; .$$

Thus, $f(x)$ belongs to a simplex of L that has the $g(p_i)$ among its vertices. But, since g is simplicial, we have

$$g(x) = \sum a_i g(p_i)$$

so that $g(x)$ lies in a simplex containing $f(x)$.

<u>II.50. Theorem</u>: For each continuous mapping f from $\underline{K}$ to $\underline{L}$, there is a barycentric subdivision $K^{(k)}$ of K that is star-related to L relative to f.

<u>Proof</u>: We note first that the star sets of the vertices v of L form an open covering of $\underline{L}$: it is clear from Remark II.6 and the definition of the star sets that $\underline{L}$ is covered. That each $\underline{*v}$ is open in $\underline{L}$ follows from the fact that $\underline{*v}$ is the complement in $\underline{L}$ of the compact set $\underline{M}$, where M is the subcomplex of L composed of simplexes of L which do not have v as a vertex.

Thus, since $\underline{L}$ is a compact metric space, there is a positive number e (the <u>Lebesgue</u> <u>number</u> of the open covering) such that any subset of $\underline{L}$ of diameter less than e is contained in some $\underline{*v}$.

Since $\underline{K}$ is compact, f is uniformly continuous and hence there is a positive number c such that

$|x - y| < c$ implies $|f(x) - f(y)| < e$.

By Theorem II.19, there is a barycentric subdivision $K^{(k)}$ of K with mesh less than $c/2$, so that if p is any vertex of $K^{(k)}$, diam $(\underline{*p}) < c$.

Thus for each vertex $p, f(\underline{*p})$ has diameter less than e , and is therefore contained in $\underline{*v}$ for some vertex v of L .

Thus, combining Theorems II.49 and II.50, it follows that any continuous mapping between polyhedra has a simplicial approximation relative to suitable triangulations of the polyhedra.

<u>II.51. Lemma</u>: Let $K = B(s^{n+1})$ be the complex of Example II.36, and let $K^{(k)}$ be a barycentric subdivision of K . Then the integral homology group $H_n(K^{(k)})$ is infinite cyclic.

<u>Proof</u>: It suffices to show that $d(z)$ is a generating cycle for $Z_n(K')$, where z is the fundamental cycle for $Z_n(K)$ and d is the chain derivation mapping. The proof is by induction on the dimension of K .

If K is 0-dimensional, then $K' = K$, $d(z) = z$ and the assertion is obviously true.

Assume the conclusion true for $\dim(K) = n - 1$.

Any n-simplex of K' is of the form $[\dot{s}_i^0 \ldots \dot{s}_j^n]$, with $s_i^0 < \ldots < s_j^n$. Consequently any n-chain on K' is of the form

$$c_n' = \sum (c_{n-1}')_i \dot{s}_i^n$$

where c_{n-1}' is an (n-1)-chain on $[B(s_i^n)]'$, the barycentric subdivision of the complex consisting of the proper faces of s_i^n.

If c_n' is a cycle on K' then

$$0 = b(c_n') = \sum_i b[(c_{n-1}')_i \dot{s}_i^n] = \sum_i [(c_{n-1}')_i - b(c_{n-1}')\dot{s}_i^n]$$

$$= \sum (c_{n-1}')_i - \sum (b(c_{n-1}')_i \dot{s}_i^n \ .$$

Since no simplex of $(c_{n-1}')_i$ contains the vertex $\dot{s}_i^n$ it follows that $b(c_{n-1}')_i = 0$ and $(c_{n-1}')_i$ is an (n-1)-cycle on $[B(s_i^n)]'$. By the induction hypothesis, $(c_{n-1}')_i = g_i d(z_{n-1}^i)$ where z_{n-1}^i is the fundamental cycle on $B(s_i^n)$. Referring to Example II.36, we see that z_{n-1}^i is of the form $b(s_i^n)$, so that we have

$$c_n' = \sum_i g_i d(b(s_i^n))\dot{s}_i^n$$

$$= \sum g_i (-1)^n \dot{s}_i^n d(b(s_i^n))$$

$$= (-1)^n \sum g_i d(s_i^n)$$

$$= d((-1)^n \sum g_i s_i^n) \ .$$

Since c_n' is a cycle, so is $d'c_n'$, where d' is an inverse of d, and

$$d'c'_n = d'[(-1)^n d(\sum g_i s_i^n)]$$

$$= (-1)^n \sum g_i s_i^n \quad .$$

Since z is the fundamental cycle on K ,

$$(-1)^n \sum g_i s_i^n = gz$$

for some integer g . Thus,

$$c'_n = d((-1)^n \sum g_i s_i^n) = gd(z) \quad ,$$

and it follows that $Z_n(K')$ is infinite cyclic.

Repeated application of the lemma shows that the successive barycentric subdivisions of $K = B(s^{n+1})$,

$$K = K^{(0)} \quad , \quad K^{(1)} \quad ,\ldots, \quad K^{(k)} \quad ,\ldots$$

have integral n-th homology groups $H_n(K^{(i)})$ which are infinite cyclic; we denote the fundamental n-cycle on $K^{(i)}$ by z^i , $i = 0,1,\ldots$.

We are now ready to define the Brouwer degree of a continuous mapping of one n-sphere into another. In the following, S,T are n-spheres, f is a continuous mapping from S into T , K is a triangulation of S whose Euclidean realization is a complex $B(s^{n+1})$, and L is a similar triangulation of T .

<u>II.52. Definition</u>: Let $K^{(k)}$ be a barycentric subdivision of K which is star related to L relative

to f , and let g be a simplicial approximation to f , relative to $K^{(k)}$ and L . If g also denotes the chain mapping induced by the simplicial mapping g , and w denotes the fundamental cycle on L , we have

$$g(z^k) = nw$$

for some integer n . This integer n is defined to be the <u>Brouwer degree of f</u> , written $\deg(f)$.

<u>II.53. Theorem</u>: $\deg(f)$ is well-defined.

<u>Proof</u>: We show that $\deg(f)$ is independent of the variable elements of its definition.

<u>(1) $\deg(f)$ is independent of the simplicial approximation g</u> : Let $\{p\}$ and $\{v\}$ be the vertices of $K^{(k)}$ and L , respectively. For a given p there may be more than one v for which $f(\underline{*p})$ is contained in $\underline{*v}$, and hence more than one simplicial approximation to f relative to $K^{(k)}$ and L . It suffices to show that $\deg(f)$ is unaffected when the value of g is changed at a single vertex p of $K^{(k)}$ to obtain a new simplicial approximation g' , since any simplicial approximation may be obtained from g by a finite number of such changes.

Thus, let $g(p) = v_i$, $g'(p) = v_j$,

$v_i \neq v_j$, and let $g(p_i) = g'(p_i)$ on all other vertices p_i of $K^{(k)}$. Let $s^n = [pp_0 \ldots p_{n-1}]$ be a nondegenerate n-simplex of $K^{(k)}$; then p is not a vertex of $s^{n-1} = [p_0 \ldots p_{n-1}]$ and we have

$$g(s^{n-1}) = g'(s^{n-1}) = t$$

where $t = [v_q \ldots v_m]$ is a simplex, possibly degenerate, in L . Since s is a simplex of $K^{(k)}$, the intersection of the star sets of the vertices is nonempty, and we have

$$f(\underline{*p} \cap (\cap \underline{*p_i})) \subset f(\underline{*p}) \cap (\cap f(\underline{*p_i})) \subset \underline{*v_i} \cap \underline{*v_j} \cap \underline{*v}_q \cap \ldots \cap \underline{*v}_m .$$

Therefore the latter intersection is nonempty, and it follows that $v_i v_j t$ is a simplex of L.

Now, if t is an m-simplex it is clear that $m \leq n - 1$; if $m < n - 1$ then t is degenerate, while if $m = n - 1$ then $v_i v_j t$ is an (n+1)-simplex in an n-complex, and must therefore be degenerate. In either case $v_i v_j t$ is zero as a chain and we have

$$0 = b(v_i v_j t) = v_j t - v_i t + v_i v_j b(t) .$$

This may also be written in the form

$$v_j g(s^{n-1}) - v_i g(s^{n-1}) + v_i v_j g(b(s^{n-1})) = 0 \quad (*) .$$

The fundamental cycle z^k can be written as a sum of chains

$$z^k = c_n + d_n$$

where each simplex in c_n has p as a vertex and no simplex in d_n has p as a vertex. c_n can then

be written as pc_{n-1} for an appropriate (n-1)-chain c_{n-1} . We then have

$$0 = b(z^k) = c_{n-1} - pb(c_{n-1}) + b(d_n)$$

which implies that $b(c_{n-1}) = 0$ since neither c_{n-1} nor $b(d_n)$ has a simplex with vertex p . Thus, applying (*) to the chain c_{n-1} ,

$$\begin{aligned} 0 &= v_j g(c_{n-1}) - v_i g(c_{n-1}) + v_i v_j b(c_{n-1}) \\ &= v_j g(c_{n-1}) - v_i g(c_{n-1}) \\ &= g'(pc_{n-1}) - g(pc_{n-1}) \quad , \end{aligned}$$

and we have $g'(c_n) = g(c_n)$.

Consequently, since no simplex of d_n has vertex p and hence $g(d_n) = g'(d_n)$, we have

$$\begin{aligned} g'(z^k) &= g'(c_n) + g'(d_n) = g(c_n) + g(d_n) \\ &= g(z^k) \quad . \end{aligned}$$

Therefore g' and g give the same value for $\deg(f)$.

<u>(2) deg(f) does not depend on the barycentric subdivision $K^{(k)}$ of K</u> : Note that if $K^{(p)}$, $K^{(m)}$ are two subdivisions of K they have a common subdivision $K^{(s)}$, so that it suffices to show that the subdivision $K^{(k+1)}$ yields the same value for $\deg(f)$ as does $K^{(k)}$.

Let d' be an inverse of the chain derivation d on $K^{(k)}$. Since $K^{(k)}$ is star-related to L relative to f , it follows that $K^{(k+1)}$ is also and gd' is simplicial approximation to f relative to

$K^{(k+1)}$ and L . Thus, if $n' = \deg(f)$ defined using the subdivision $K^{(k+1)}$, we have

$$n'w = g(d'(z^{k+1})) = g(d'(d(z^k))) = g(z^k) = nw \quad .$$

Consequently $n' = n$, and hence any suitably fine subdivision of K may be used in defining $\deg(f)$.

<u>(3)</u> <u>deg(f)</u> <u>is</u> <u>independent</u> <u>of</u> <u>the</u> <u>complexes</u> <u>K , L</u> : We first show that a subdivision of L yields the same value for $\deg(f)$; it clearly suffices to consider only the first barycentric subdivision L' .

Let $K^{(m)}$ be a subdivision of K that is star-related to L' relative to f . Certainly $K^{(m)}$ is star-related to L , as well as to L' , relative to f . Let $n = \deg(f)$ as determined using $K^{(m)}$ and L , and let $n' = \deg(f)$ as computed using $K^{(m)}$ and L' , using the simplicial approximations g and g' respectively. Then with w the fundamental cycle on L , and hence $d(w)$ the fundamental cycle on L' , we have

$$g(z^m) = nw \quad , \qquad g'(z^m) = n'd(w) \quad .$$

As before, $d'g'$ is a simplicial approximation to f relative to $K^{(m)}$ and L , and hence by part (1) above,

$$d'(g'(z^m)) = nw \quad .$$

But also,

$$d'(g'(z^m)) = d'(n'd(w)) = n'd'(d(w)) = n'w \quad ,$$

so that $n' = n$.

Now let L^* be a triangulation of T, where L^* is still a $B(s^{n+1})$. Let M be a subdivision of L^* which refines L, and suppose $K^{(k)}$ is star-related to M, and hence to L, relative to f . Since M is a refinement of L, a simplicial mapping h from M to L can be defined, in an obvious way. Denoting the induced chain mapping also by h, and the fundamental cycles on L and M by w and u, respectively, we have

$$h(u) = pw$$

for some integer p . We assume L, L^* to be mutually oriented so that p is nonnegative. Also, it is clear that if g is a simplicial approximation to f relative to $K^{(k)}$ and M, then hg is a simplicial approximation to f relative to $K^{(k)}$ and L . Thus, if $\deg(f) = m$ relative to M and $\deg(f) = n$ relative to L, we have

$$nw = h(g(z^k)) = h(mu) = mpw \quad ,$$

so that $n = mp$.

Similarly, taking a subdivision of L which refines L^*, we have, for some integer q

$$m = qn \quad .$$

Hence, if one of m,n is zero, so is the other.

Otherwise, $pq = 1$, and we have $p = q = 1$. Therefore, $m = n$, and $\deg(f)$ is independent of the particular triangulation L .

Now let K^* be a triangulation of S , which is also a $B(s^{n+1})$, and let $K^{(k)}$ be star-related to L relative to f . Subdivide K^* to obtain a K_1^* , which refines $K^{(k)}$. Then there is a simplicial mapping, defined in an obvious way, from K_1^* to $K^{(k)}$; we denote this mapping and its induced chain mapping by h . Thus, denoting the fundamental cycle on $K^{(k)}$, K_1^* by z^k, z^* , respectively, we have

$$h(z^*) = pz^k$$

for some integer p ; we assume K^* and K to be mutually oriented so that p is nonnegative.

If g is a simplicial approximation to f relative to $K^{(k)}$ and L , then since K_1^* refines $K^{(k)}$, gh is a simplicial approximation to f relative to K_1^* and L . Consequently, we have

$$mw = g(h(z^*)) = g(pz^k) = pnw$$

where m,n is the degree of f defined relative to K_1^* , $K^{(k)}$, respectively. Similarly, for some integer q ,

$$nw = qmw \quad .$$

Thus as before, $n = m$ and $\deg(f)$ is independent of the complex K .

II.54. Remarks: (i) Returning to the definition of $\deg(f)$, suppose the n-simplexes s_i of $K^{(k)}$ and the n-simplexes t_j of L are so oriented that they appear with positive sign in z^k , w ; i.e.,

$$z^k = \sum s_i \quad , \quad w = \sum t_j \quad .$$

For any simplex t_j there will be, say, p_j simplexes s'_{ij} of $K^{(k)}$ such that $g(s'_{ij}) = +t_j$, $i = 1,\dots,p_j$, and q_j simplexes s''_{ij} such that $q(s''_{ij}) = -t_j$, $i = 1,\dots,q_j$; some simplexes of $K^{(k)}$ may also be collapsed by g . Thus we can write

$$Z^k = \sum_j \sum_{i=1}^{p_j} s'_{ij} + \sum_j \sum_{i=1}^{q_j} s''_{ij} + c \quad ,$$

where c is an n-chain such that $g(c) = 0$.

Hence

$$n\sum t_j = nw = g(z^k) = \sum_j(\sum_i g(s'_{ij}) + \sum_h g(s''_{hj}))$$

$$= \sum_j (p_j - q_j)t_j \quad ,$$

which implies that $n = p_j - q_j$ for all j . Therefore $p_j - q_j$ is constant, and the degree of f may be intuitively regarded as the algebraic number of times that f "wraps" S around T .

(ii) It is evident from part (3) of the proof of Theorem II.53 that $\deg(I) = 1$, where I is the identity mapping on the sphere.

If h is a homeomorphism between n-spheres, then from the product theorem below it follows that

$$\deg(h)\deg(h^{-1}) = \deg(hh^{-1}) = \deg(I) = 1 ,$$

so that the degree of a homeomorphism is ± 1.

If f is a constant mapping, then any simplicial approximation collapses the simplexes of $K^{(k)}$, so that $\deg(f) = 0$.

<u>II.55. Theorem</u>: Let S, T, and U be n-spheres, and suppose $f : S \to T$ $g : T \to U$ are continuous. Then $\deg(gf) = \deg(g)\deg(f)$.

<u>Proof</u>: Let K, L, M be triangulations of S, T, U, respectively, and let $K^{(k)}$, $L^{(m)}$ be subdivisions such that $K^{(k)}$ is star-related to $L^{(m)}$ relative to f, $L^{(m)}$ is star-related to M relative to g. Then $K^{(k)}$ is star-related to M relative to gf. Also, if F is a simplicial approximation to f relative to $K^{(k)}$ and $L^{(m)}$, and G is a simplicial approximation to g relative to $L^{(m)}$ and M, then it is evident that GF is

a simplicial approximation to gf relative to $K^{(k)}$ and M. Thus denoting the fundamental cycles on $K^{(k)}$, $L^{(m)}$, M by z^k, w^m, u, respectively, we have

$$\deg(gf)u = (GF)(z^k) = G(F(z^k)) = G(\deg(f)w^m)$$
$$= \deg(f)\deg(g)u \quad .$$

Therefore $\deg(gf) = \deg(g)\deg(f)$, which was to be proved.

<u>II.56. Theorem</u>: Let f,g be continuous mappings of S into T. If f and g are homotopic, then $\deg(f) = \deg(g)$.

<u>Proof</u>: If f and g are homotopic, there is a continuous mapping $h : S \times [0,1] \to T$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for x in S. Let a be the Lebesgue number of the (finite) open covering $0 = \{\underline{*v} : v$ is a vertex of $L\}$ of $\underline{L}$. Since $S \times [0,1] = \underline{K} \times [0,1]$ is compact, h is uniformly continuous, so that there is a $\delta > 0$ with the property that if A, B are subsets of S and $[0,1]$, respectively, with diameter $< \delta$, then $h(A \times B)$ has diameter $< a$.

Let $K^{(p)}$ be a subdivision of K with mesh $< \delta/2$ and let $0 = t_0 < \ldots < t_k = 1$ be a

partition of $[0,1]$ such that $t_i - t_{i-1} < \delta$. Then each $\underline{*p_i}$, p_i a vertex in $K^{(p)}$, and each open interval (t_{i-1} , t_i) has diameter $< \delta$, which implies that each set $h(\underline{*p_i} \times (t_{i-1} , t_i)) = P_i$ has diameter $< a$ and hence is contained in $\underline{*v}$ for some vertex v of L .

For each t in $[t_{i-1} , t_i]$, consider the mapping $h_t = h/S \times \{t\}$ (i.e., the restriction of h to $S \times \{t\}$) from S into T . In determining $\deg(h_t)$, the simplicial approximation, g , can be defined by $g(p_i ; t) = v_j$, where P_i is contained in $\underline{*v_j}$. Using the same g for each t in $[t_{i-1} , t]$, it follows that $\deg(h_t)$ is constant on each interval $[t_{i-1} , t_i]$. Since these intervals overlap, $\deg(h_t)$ is constant on $[0,1]$. But $h_0 = f$ and $h_1 = g$, and hence $\deg(f) = \deg(g)$.

We note that the converse of the above theorem is true: If two continuous mappings from an n-sphere S into an n-sphere T have the same degree, then they are homotopic. (Hopf [1]). Thus the degree completely characterizes the homotopy classes of mappings of n-spheres.

Having defined the degree of a mapping and

established some of its basic properties, we are now in a position to apply it to a proof of the Brouwer fixed point theorem. The essential part of the proof is isolated in the following theorem, of interest in its own right.

<u>II.57. Definition</u>: A <u>retraction</u> of a topological space X onto a subset A of X is a continuous mapping $r : X \to A$ such that $r(a) = a$ for all a in A.

<u>II.58. Theorem</u>: There is no retraction of an n-cell onto its boundary $(n > 0)$.

<u>Proof</u>: Without loss of generality, we take the n-cell to be $C = \{(x_1,\ldots,x_n) : \sum x_i^2 \le 1\}$ in E^n, whose boundary is the (n-1)-sphere $S = \{(x_1,\ldots,x_n) : \sum x_i^2 = 1\}$.

Suppose there is a retraction r of C onto S, and consider the mappings h_t on S into S given by

$$h_t(x) = r((1-t)x) \quad , \quad 0 \le t \le 1 \ .$$

Then $h_1(x) = r(0)$, so that h_1 is a constant mapping, and hence $\deg(h_1) = 0$. On the other hand,

$h\ (x) = r(x) = x$, so h is the identity on S , and $\deg(h\ ) = 1$.

Since the mapping $h : S \times [0,1] \to S$ defined by

$$h(x,t) = h_t(x)$$

is clearly a homotopy, we have

$$0 = \deg(h_1) = \deg(h_0) = 1 \quad ,$$

which is impossible. Thus we conclude that no retraction r exists.

<u>II.59. Theorem</u> (Brouwer [3]) Every continuous self-mapping of an n-cell has a fixed point.

<u>Proof</u>: As in the previous theorem, we take $C = \{x : \sum x_i^2 \leq 1\}$. Suppose the theorem false, and let f be a continuous self-mapping of C with no fixed point. For each x in C , let $L(x)$ be the ray originating at $f(x)$ which passes through x . Since there is no fixed point, each pair $(x,f(x))$ determines a unique ray, and hence a unique point y_x in the intersection $L(x) \cap (S - f(x))$. Define $r(x) = y_x$. Then r is continuous, and $r(x) = x$ for each x in S . Thus r is a retraction of C onto its boundary S , which contradicts the previous theorem. Consequently f

must have a fixed point.

<u>II.60. Remark</u>: We note that the no-retraction theorem, Theorem II.58, is actually equivalent to the Brouwer fixed point theorem. For, assume Theorem II.59 to hold, and suppose a retraction r of C onto S exists. Let h be the mapping on S which takes a point of S into the point diametrically opposite it. Then the composite mapping hr is a continuous self-mapping of C which obviously has no fixed point. Since this contradicts the Brouwer fixed point theorem, we conclude that no retraction r exists.

Since the publication of Brouwer's basic papers, much effort has been made to establish his results by methods other than those of Algebraic topology. Thus, for example, E. Heinz [1], has, by means of a volume integral, defined the degree of a continuous mapping defined in the closure of a bounded open set in n-dimensional space; the main properties of the degree are then derived and a proof of the fixed point theorem is given. An analytic proof of the Brouwer fixed point theorem which does not

involve the concept of degree of a mapping is also given in Dunford and Schwartz [1], pp. 467 - 470. Other analytic proofs of the fixed point theorem have been given by Birkhoff and Kellogg [1], and by Seki [1].

We conclude this chapter with the most elementary of the alternate proofs of the Brouwer fixed point theorem, due to Knaster, Kuratowski, and Mazurkiewicz [1].

II.61. Lemma: Let $s = [p_0 \ldots p_k]$ be partitioned into subsimplexes in such a way that the intersection of any two subsimplexes is empty, or is a face of each. With each vertex v of the subdivision, associate an integer $n(v)$ in such a way that if v belongs to the face $[p_{i_0} \ldots p_{i_m}]$ of s, then $n(v)$ is one of the numbers $i_0, \ldots, i_m$. Then there is at least one subsimplex at whose vertices n takes on all values $0, 1, \ldots, k$. Such a simplex will be called a representative simplex, and the number R of representative simplexes is odd.

Proof: Let t be a simplex in the subdivision of s. A face t^* of t will be called a representative face, or, briefly, an r-face, of t if n assumes the values

$0,1,\ldots,k-1$ on the vertices of t^*. The proof of the lemma is by induction on k.

For $k = 0$, the lemma is trivial, so we assume its validity for $k-1$. Letting

N = number of r-faces on the boundary of s

$A(t)$ = number of r-faces of a subsimplex t,

we first show that $R \equiv N \pmod 2$; i.e., R and N are both odd or both even. It is clear that if t is a representative simplex then $A(t) = 1$. If t is not representative then $A(t) = 0$ or 2 depending on whether n omits one of the values $0,1,\ldots,k-1$ or the value k as it ranges over the vertices of t. It follows that

$$R \equiv \sum A(t) \pmod 2,$$

where the summation is taken over all subsimplexes t of the partition. From the manner in which s is subdivided, each r-face is represented once or twice in the summation, depending on whether or not it lies on the boundary of s. Consequently,

$$N \equiv \sum A(t) \pmod 2,$$

and hence

$$R \equiv N \pmod 2.$$

Consider now any $(k-1)$-face s^* of s different from $[p_0 \ldots p_{k-1}]$. Clearly $s^* = [p_0 \ldots \hat{p}_i \ldots p_k]$ for

some $i = 0,1,\ldots,k-1$, so that there is no point q in s^* for which $n(q) = i$. Consequently s^* does not contain an r-face. In other words, all r-faces in the boundary of s are contained in the face $[p_0 \ldots p_{k-1}]$, and therefore N denotes the number of r-faces contained in this face. Since the r-faces in $[p_0 \ldots p_{k-1}]$ are the representative simplexes in the subdivision of $[p_0 \ldots p_{k-1}]$, it follows from the induction hypothesis that N, and hence R by the above congruence, is odd.

<u>II.62. Lemma</u>: Let $A_0,\ldots,A_k$ be closed sets such that every q-face $[p_{i_0} \ldots p_{i_q}]$ of $s = [p_0 \ldots p_k]$ is contained in the union $A_{i_0} \cup \ldots \cup A_{i_q}$. Then the intersection

$$\cap \{A_i : i = 0,\ldots,k\}$$

is nonempty.

<u>Proof</u>: For a fixed $m > 0$, let $s^{(m)}$ be a subdivision of s such that each subsimplex has diameter $< 1/m$. Let w be a vertex in the subdivision, and let $[p_{i_0} \ldots p_{i_q}]$ be the unique face of s which has w as an inner point. By hypothesis,

$$[p_{i_0} \ldots p_{i_q}] \subset A_{i_0} \cup \ldots \cup A_{i_q} ,$$

hence there is an i_j , $0 \leq j \leq q$ for which $w \varepsilon A_{i_j}$. If we set $n(w) = i_j$, then n satisfies the preceding lemma; we can write $w \varepsilon A_{n(w)}$.

By the lemma, there is a representative simplex, which we denote by $[w_0^m, \ldots, w_k^m]$, where we set $n(w_i^m) = i$. We then have $w_i^m \varepsilon A_i$.

Consider the sequence $\{w_0^m\}$; we may assume this sequence is convergent (otherwise replace it by a convergent subsequence), say to a . Since the diameter of the subsimplexes approaches zero as m increases, it follows that

$$\lim_{m \to \infty} w_i^m = a \quad , \qquad i = 0, \ldots, k \quad .$$

Thus, since the sets A_i are closed, we have $a \varepsilon A_i$ for each $i = 0, \ldots, k$, and the intersection is nonempty.

<u>II.63. Theorem</u>: (Brouwer [3]) A continuous self-mapping f of a k-simplex $s = [p_0 \ldots p_k]$ has a fixed point.

<u>Proof</u>: Let A_i be the set of points x of s for which $c_i' \leq c_i$, where c_i' , c_i , $i = 0, \ldots, k$ are the barycentric coordinates of $f(x)$, x , respectively, with respect to the vertices p_i of s . By continuity of f , the sets A_i are closed; we now show that

$A_0, \ldots, A_k$ satisfy the preceding lemma.

To this end, let $s^q = [p_{i_0} \ldots p_{i_q}]$ be a q-face of s, and suppose some point $x = c_{i_0} p_{i_0} + \ldots + c_{i_q} p_{i_q}$ is not contained in any one of the sets A_{i_j}, $j = 0,1,\ldots,q$. Then we have $c'_{i_j} > c_{i_j}$, $j = 0,1,\ldots,q$, which in turn implies

$$c'_{i_0} + \ldots + c'_{i_q} > c_{i_0} + \ldots + c_{i_q} = 1$$

which is impossible. Thus, $s^q \subset \cup \{A_{i_j} : j = 0,\ldots,q\}$ so that the lemma applies.

Let

$$x = c_0 p_0 + \ldots + c_k p_k \in \cap A_i \;;$$

then for $f(x) = c'_0 p_0 + \ldots + c'_k p_k$ we have

$$c'_i \leq c_i \;, \quad i = 0,1,\ldots,k \;.$$

This, together with

$$1 = c'_0 + \ldots + c'_k \leq c_0 + \ldots + c_k = 1$$

implies that $c'_i = c_i$, $i = 0,1,\ldots,k$. That is, $f(x) = x$ and the theorem is proved.

CHAPTER III

THE FIXED-POINT THEOREMS OF LEFSCHETZ, SCHAUDER, AND TYCHONOFF

Following the work of Brouwer, the next major contribution to fixed point theory was made by S. Lefschetz in several papers published during the period 1923 - 1942. In the first of these (Lefschetz [2]), a formula is given which provides a sufficient condition for the existence of fixed points for a continuous self-mapping of a compact manifold without boundary. The successive papers extended the validity of the fixed point formula to wider classes of spaces (see Lefschetz [3], [4], [5], [6]).

We shall define the Lefschetz number and give the proof of the Lefschetz fixed point theorem for continuous self-mappings of polyhedra. We begin by showing how, for complexes K, L, a continuous mapping of $\underline{K}$ into $\underline{L}$ can be used to define a homomorphism between the homology groups of K and L.

<u>III.1. Definition</u>: Let K, L be complexes, G an abelian group, and suppose f is a continuous mapping of <u>K</u> into <u>L</u>. Further let $K^{(m)}$ be a subdivision of K which is star-related to L relative to f (Theorem II.50), and let g be a simplicial approximation to f relative to $K^{(m)}$ and L (Theorem II.49). Denoting by g^* the homomorphism induced by g,

$$g^* : H_p(K^{(m)},G) \to H_p(L,G) \quad ,$$

and by d^m* the homomorphism induced by the m-th iterate of chain derivation d,

$$d^m* : H_p(K,G) \to H_p(K^{(m)},G) \quad ,$$

we define

$$f^* = (gd^m)^* = g^*d^m* \quad .$$

f^* is called the <u>homomorphism of $H_p(K,G)$ into $H_p(L,G)$ induced by f</u>.

We must now show that f^* is well-defined, i.e., the definition of f^* is independent of the subdivision $K^{(m)}$ and the simplicial approximation g. In order to do this, we first establish a sufficient condition for two chain mappings to induce the same homomorphism of the homology groups.

III.2. Definition: Let K, L be complexes, G an abelian group, and suppose f, g are simplicial mappings of $\underline{K}$ into $\underline{L}$. Denote the induced chain mappings of $C_p(K,G)$ into $C_p(L,G)$ also by f, g.

(i) f and g are said to be close if, whenever $[a_0 \ldots a_p]$ is a simplex in K, then $f(a_0),\ldots,f(a_p),g(a_0),\ldots,g(a_p)$ are (not necessarily distinct) vertices of a simplex in L.

(ii) f and g are algebraically homotopic if there is a homomorphism

$$h : C_p(K,G) \to C_{p+1}(L,G)$$

such that for any chain c in $C_p(K,G)$,

$$f(c) - g(c) = b(h(c)) + h(b(c)) ,$$

where b is the boundary operator.

III.3. Lemma: Let K, L, f, g be as in the preceding definition. Then if f and g are close, they are algebraically homotopic.

Proof: We proceed by induction on p to show that, for an elementary p-chain s, $s \in K$, there is a homomorphism h such that

$$f(s) - g(s) = b(h(s)) + h(b(s)) .$$

For $p = 0$, s is a vertex a of K. Define

$$h(a) = [g(a)f(a)] \quad \text{if} \quad f(a) \neq g(a)$$

$$h(a) = 0 \quad \text{if} \quad f(a) = g(a) \quad ,$$

and extend h linearly to arbitrary 0-chains.
Then h is a homomorphism,

$$h : C_0(K,G) \to C_1(L,G) \quad ,$$

and

$$f(a) - g(a) = b[g(a)f(a)] = b(h(a)) = b(h(a)) + h(b(a)) \; .$$

Since f and g are close, for any s in K it follows that $f(s)$ and $g(s)$ are contained in some simplex t of L. We assume that, for (p-1)-chains u, h has been defined so that $h(u)$ is contained in t. Now, for an elementary p-chain s, let

$$w = f(s) - g(s) - h(b(s)) \quad .$$

It follows from the induction hypothesis and the fact that f and g are close that w is a (possibly degenerate) chain on the p-faces of t.
Also,

$$b(w) = b(f(s)) - b(g(s)) - b(h(b(s))) = f(b(s))$$
$$- g(b(s)) - b(h(b(s)))$$
$$= b(h(b(s))) + h(b(b(s))) - b(h(b(s))) = 0 \quad ,$$

since, by the induction hypothesis, $f(b(s)) - g(b(s)) = b(h(s)) + h(b(s))$ for the (p-1)-chain $b(s)$.
Thus, w is a cycle. Furthermore, if v is a vertex

of t, we may write

$$w = vc + c'$$

where c, c' are p-chains on the p-faces of t such that no simplex in either c or c' has v as a vertex. Thus,

$$0 = b(w) = c - vb(c) + b(c') \quad ,$$

which implies that $c = -b(c')$ and hence

$$w = -vb(c') + c' = b(vc') \quad .$$

We now define $h(s) = vc'$. It then follows that

$$w = b(vc') = b(h(s)) = f(s) - g(s) - h(b(s)) \quad ;$$

i.e.,

$$f(s) - g(s) = b(h(s)) + h(b(s)) \quad ,$$

which was to be shown.

<u>III.4. Theorem</u>: Let f and g be algebraically homotopic chain mappings of $C_p(K,G)$ into $C_p(L,G)$. Then the induced homomorphisms f^*, g^* of $H_p(K,G)$ into $H_p(L,G)$ are identical.

<u>Proof</u>: Let z be a p-cycle on K; then

$$f(z) - g(z) = b(h(z)) + h(b(z)) = b(h(z))$$

and hence $f(z) - g(z)$ is a p-boundary. It then follows from the definition of the induced homomorphism that $f^* = g^*$.

<u>III.5. Theorem</u>: Let K, L be complexes, and let f be a continuous mapping of $\underline{K}$ into $\underline{L}$. The induced homomorphism f* of $H_p(K,G)$ into $H_p(L,G)$ is well defined, and for $f : \underline{K} \to \underline{L}$, $f' : \underline{L} \to \underline{M}$ we have

$$(f'f)^* = f'^*f^* .$$

<u>Proof</u>: We first show that if $K^{(m)}$, $K^{(n)}$ are two subdivisions of K with associated simplicial approximations g , g' , then $(gd^m)^* = (g'd^n)^*$.

Consider first the case $m = n$. From part (1) of the proof of Theorem II.53, it follows that g and g' are close. Consequently, they are algebraically homotopic, and it follows that $g^* = g'^*$. (This also shows that the definition of f* does not depend on the particular simplicial approximation).

Next, let $n = m + p$, $p > 0$. Note that $(d^{p+m})^* = d^{p*}d^{m*}$, so that it suffices to show $g^* = g'^*d^{p*}$, where $d^p : C(K^{(m)},G) \to C(K^{(m+p)},G)$. By the first case considered, it does not matter which simplicial approximation is used, relative to a given pair of complexes. Thus, if I denotes the identity mapping on $\underline{K}^{(p+m)} = \underline{K}^{(m)}$, let i be a simplicial approximation to I relative to $K^{(m+p)}$

and $K^{(m)}$, and choose $g' = gi$. We then have

$$g'^{*}d^{p*} = (gi)^{*}d^{p*} = g^{*}i^{*}d^{p*} = g^{*}(id^{p})^{*} .$$

But i is clearly an inverse to d^{p}, and hence id^{p} induces the identity homomorphism.

Consequently,

$$g^{*} = g'^{*}d^{p*} .$$

To show that $(f'f)^{*} = f'^{*}f^{*}$, subdivide L m times and K n times, where m, n are sufficiently large so that g' is a simplicial approximation to f' relative to $L^{(m)}$ and g is a simplicial approximation to f relative to $K^{(n)}$ and $L^{(m)}$. Let $i : \underline{L^{(m)}} \to \underline{L}$ be a simplicial approximation to the identity on $\underline{L^{(m)}} = \underline{L}$, relative to $L^{(m)}$ and L. Define $h = ig$. Then

$$f^{*} = (hd^{n})^{*} = i^{*}g^{*}d^{n*} ,$$

$$f'^{*} = g'^{*}d^{m*} = (g'd^{m})^{*} ,$$

and we have

$$(f'f)^{*} = (g'gd^{n})^{*} = g'^{*}g^{*}d^{n*} ,$$

$$f'^{*}f^{*} = g'^{*}d^{m*}i^{*}g^{*}d^{n*} .$$

But $d^{m}i$ is a simplicial approximation to the identity on $\underline{L^{(m)}} = \underline{L}$ relative to $L^{(m)}$ and $L^{(m)}$; hence $d^{m*}i^{*} = (d^{m}i)^{*}$ is the identity homomorphism, and we have

$$f'^{*}f^{*} = g'^{*}g^{*}d^{n*} = (f'f)^{*} .$$

<u>III.6. Theorem</u>: Let K and L be complexes, and let f, g be continuous mappings of $\underline{K}$ into $\underline{L}$. If f and g are homotopic, then $f^* = g^*$.

<u>Proof</u>: The proof of this theorem is completely analogous to the proof of Theorem II.56.

In the following discussion, all groups $C_p(K,G),\ldots,H_p(K,G)$ will be the <u>rational</u> groups, that is, the coefficient group G will be taken to be the field Q of rational numbers. As in the case of the integral groups, we shall write $C_p(K),\ldots,$ $H_p(K)$ for $C_p(K,Q)$, etc. We shall further abbreviate the notation to $C_p,\ldots,H_p$ when it is clear which complex is concerned. Note that since Q is a field, the groups $C_p,\ldots,H_p$ are linear spaces over Q.

<u>III.7. Definition</u>: (i) Let $K = \{s_i^p : p = 0,\ldots,n\,;$ $i = 1,\ldots,a(p)\}$ be a complex and $F : C_p(K) \to C_p(K)$ a chain mapping. An element s_i^p of K will be called a <u>fixed element under F</u> if, in the chain

$$F(s_i^p) = \sum_{j=1}^{a(p)} a_{ij}^{(p)} s_j^p \quad , \qquad a_{ij}^{(p)} \in Q \quad ,$$

the coefficient $a_{ii}^{(p)}$ of s_i^p is non-zero.

(ii) Let K^* be a simplicial

subdivision of a complex K, and $g : \underline{K^*} \to \underline{K}$ a simplicial mapping. A simplex $*s_i^p$ of K^* will be called a <u>fixed simplex under g</u> if $*s_i^p$ is not collapsed by g, and $*s_i^p$ is contained in $g(*s_i^p)$.

<u>III.8. Remark</u>: We note here for future use that if $g' : C_p(K^*) \to C_p(K)$ is a chain mapping induced by a simplicial mapping $g: K^* \to K$, and d is the chain derivation mapping, then a fixed element $*s_i^p$ under dg' is a fixed simplex under g. For, $g'(*s_i^p)$ is a simplex t_j^p in K (since g is simplicial), and if $*s_i^p$ appears in $d(t_j^p)$ with non-zero coefficient, then $*s_i^p \subset t_j^p$ and g does not collapse $*s_i^p$.

<u>III.9. Definition</u>: Let K and F be as in the preceding definition. The <u>trace invariant of F</u>, denoted $T(F)$ is defined by

$$T(F) = \sum_{p=0}^{n} (-1)^p \quad \text{trace} \quad (a_{ij}^{(p)}) \quad ,$$

where $(a_{ij}^{(p)})$ is the matrix of the homomorphism F with respect to a given base for $C_p(K)$, and

$$\text{trace} \quad (a_{ij}^{(p)}) = \sum_{i=1}^{a(p)} a_{ii}^{(p)} \quad .$$

<u>III.10. Remark</u>: That $T(F)$ is well defined follows from the fact that the trace is independent of the

particular choice of basis used to define $(a_{ij}^{(p)})$. That is, if $(a_{ij}^{(p)})$ is the matrix of F with respect to a base $\{(c_p)_i\}$ for $C_p(K)$, and $(b_{ij}^{(p)})$ is the matrix of F with respect to a second base $\{(d_p)_i\}$, then trace $(a_{ij}^{(p)})$ = trace $(b_{ij}^{(p)})$. For a proof of this, see Alexandroff and Hopf [1], page 568.

III.11. Definition: Let K be an n-dimensional complex, $h : C_p(K,Q) \to C_p(K,Q)$ a homomorphism, and $\{e_i^{(p)} : i = 1,\ldots,a(p)\}$ a base for $H_p(K,Q)$, $p = 0,1,\ldots,n$. The Lefschetz number of h , denoted by $L(h)$, is defined by

$$L(h) = \sum_{p=0}^{n} (-1)^p \ \text{trace} \ (a_{ij}^{(p)}) ,$$

where $(a_{ij}^{(p)})$ is the matrix of h with respect to $\{e_i^{(p)}\}$.

III.12. Remarks: (i) As before, trace $(a_{ij}^{(p)})$ is independent of the particular base chosen to define $(a_{ij}^{(p)})$, so that $L(h)$ is well-defined.

(ii) If K^* is a barycentric subdivision of K , and g is a simplicial mapping of K^* into K , then it follows from part (1) of the proof of Theorem III.5 that $g^* = (dg)^*$, where d is the chain derivation mapping. Consequently, we

have $L(g^*) = L((dg)^*)$.

Notation: If X, Y are linear spaces, then for a homomorphism

$$h : X \to Y ,$$

denote the trace of the matrix of h relative to a given pair of bases for X, Y by $\operatorname{tr} h[X,Y]$, or, if $X = Y$, by $\operatorname{tr} h[X]$.

III.13. Lemma: Let K be an n-dimensional complex, and for a simplicial mapping g' of $\underline{K}$ into itself, let g, g^* denote, respectively, the chain mapping on $C_p(K)$ induced by g' , and the homomorphism on $H_p(K)$ induced by g . Then we have

$$T(g) = L(g^*) .$$

Proof: Let $h' : C_p \to B_{p-1}$ be the mapping which assigns to every chain c its boundary : $h'(c) = b(c)$. Clearly h' is a homomorphism onto, with kernel Z_p , and hence induces naturally an isomorphism H between C_p/Z_p and B_{p-1} . Also, g as a chain mapping on C_p induces endomorphisms G of C_p/Z_p and G' of B_{p-1} . Furthermore, $HG = G'H$, since for any $c + Z_p$ in C_p/Z_p ,

$$\begin{aligned}(HG)(c+Z_p) &= H(g(c)+Z_p) = h'(g(c)) \\ &= b(g(c)) = g(b(c)) \\ &= G'(b(c)) = G'(h'(c))\end{aligned}$$

$$= (G'H)(c+Z_p) \quad .$$

Therefore, since H is an isomorphism, the matrices of G and G' are similar, and hence

$$\mathrm{tr}\ G'[B_{p-1}] = \mathrm{tr}\ G[C_p/Z_p] \quad .$$

Since the trace is "additive" (see Alexandroff and Hopf [1], p. 570), and since g induces G' and G, we have

$$\mathrm{tr}\ g[C_p] - \mathrm{tr}\ g[Z_p] = \mathrm{tr}\ g[B_{p-1}] \quad ;$$

or,

$$\mathrm{tr}\ g[C_p] = \mathrm{tr}\ g[Z_p] + \mathrm{tr}\ g[B_{p-1}] \quad .$$

Again by additivity,

$$\mathrm{tr}\ g^*[H_p] = \mathrm{tr}\ g[Z_p] - \mathrm{tr}\ g[B_p] \quad .$$

Multiplying the last two equations by $(-1)^p$ and summing over p from 0 to n ($B_n = B_{-1} = \{0\}$, so that $\mathrm{tr}\ g[B_n] = \mathrm{tr}\ g[B_{-1}] = 0$), we get upon subtracting

$$\sum_{p=0}^{n} (-1)^p\ \mathrm{tr}\ g[C_p] - \sum_{p=0}^{n} (-1)^p\ \mathrm{tr}\ g^*[H_p] = 0 \quad ;$$

i.e.,

$$T(g) = L(g^*) \quad .$$

<u>III.14. Lemma</u>: Let K be an n-dimensional complex, and let

$$g : C_p(K) \to C_p(K)$$

be a chain mapping. If $T(g) \neq 0$, then g has a

fixed element.

Proof: If $T(g) \neq 0$, then trace $(a_{ij}^{(p)}) \neq 0$ for some $p = 0,1,\ldots,n$, so that for some i, $a_{ii}^{(p)} \neq 0$. Consequently s_i^p is a fixed element.

III.15. Theorem (Lefschetz [1], p. 156) Let K be an n-dimensional complex, and let f be a continuous self-mapping of $\underline{K}$. If the Lefschetz number of the homomorphism induced by f on the homology groups $H_p(K,Q)$ is nonzero, then f has a fixed point.

Proof: Suppose $L(f^*) \neq 0$, and f has no fixed point. Since $\underline{K}$ is a compact metric space and the function

$$h(x) = d(x,f(x))$$

is continuous (d is a metric on $\underline{K}$), it follows that

$$e = \inf\{h(x) : x \in \underline{K}\} > 0 .$$

Without loss of generality, we assume that

$$\text{mesh } K < e/2$$

(otherwise replace K by a suitable subdivision). Let K^* be a subdivision of K, and let g be a simplicial approximation to f relative to K^* and K. Then g is homotopic to f, and $d(f(x),g(x)) < e/2$ for each x in $\underline{K^*} = \underline{K}$ (since $f(x),g(x)$ are

in the same simplex of K).

Thus, by the preceding results, we have

$$0 \neq L(f^*) = L(g^*) = L((dg)^*) = T(dg) \quad .$$

Consequently, the chain mapping dg has a fixed element s^* in K^*, which is also a fixed simplex for g. Let $x^* \varepsilon s^*$; then x^* and $g(x^*)$ belong to the same simplex of K, so that

$$d(x^*,g(x^*) < e/2 \quad .$$

But then

$$h(x^*) = d(x^*,f(x^*)) \leq d(x^*,g(x^*)) + d(g(x^*),f(x^*))$$
$$< e \quad ,$$

which contradicts the definition of e. Thus we conclude that f has a fixed point.

<u>III.16. Remark</u>: The great scope of the Lefschetz fixed point theorem is attested by the fact that it included almost all fixed point theorems known at the time of its publication. For example, we now show the Brouwer fixed point theorem, Theorem II.63, to be a special case.

Let s be a n-simplex, and let f be a continuous self-mapping of s. Let K be the complex consisting of s together with all its faces. By Example II.36, the <u>integral</u> homology groups

$H_p(K)$ are cyclic for $p = 0$, and are zero for $p = 1,\ldots,n-2$. To determine $H_n(K)$, note that there are no $(n+1)$-chains on K, so that $B_n(K) = \{0\}$. Also, any integral n-cycle z on K is of the form

$$z = ms \ , \quad m \text{ an integer},$$

since s is the only n-simplex in K.

Then

$$0 = b(z) = mb(s) \ ,$$

which implies $m = 0$. Consequently, $Z_n(K) = \{0\}$ and we have

$$H_n(K) = \{0\}/\{0\} = \{0\} \ .$$

Again by Example II.36, any integral $(n-1)$-cycle z is of the form

$$z = mb(s) = b(ms) \ , \quad m \text{ an integer} \ ,$$

so that z is also an $(n-1)$-boundary; i.e., $Z_{n-1}(K) = B_{n-1}(K)$, and we have

$$H_{n-1}(K) = \{0\} \ .$$

Now, any <u>rational</u> cycle z on K can be written in the form $z = qc$, where q is a rational number and c is an integral cycle. It follows that the rational homology groups $H_p(K,Q)$ also are cyclic for $p = 0$, and are zero for $p = 1,\ldots,n$. Thus, if f^* is the homomorphism induced by f on $H_p(K,Q)$, we have

$$\operatorname{tr} f^* [H_p(K,Q)] = 0 \quad , \quad p = 1,\ldots,n \quad .$$

Now, since $H_0(K,Q)$ is cyclic, since f^* is defined in terms of a simplicial mapping, and since any two vertices of K are homologous, we have, for $a + B_0 \in H_0(K,Q)$,

$$f^*(a + B_0) = a' + B_0 = a + B_0 \quad ,$$

where a,a' are vertices of K . Therefore, the matrix of f^* relative to a base for $H_0(K,Q)$ is the 1×1 unit matrix (1) , and we have

$$L(f^*) = \operatorname{tr}(1) = 1 \quad .$$

Consequently, by Lefschetz' theorem, f has a fixed point.

In contrast to the homological methods used by Lefschetz to establish the fixed point formula, other writers adopted simpler convexity and compactness arguments to extend the Brouwer fixed point theorem for the n-cell to certain subsets of linear spaces. The most significant results in this direction were obtained by J. Schauder and A. Tychonoff, and it is to these that we now turn our attention.

In 1922, G. D. Birkhoff and O. D. Kellogg [1] proved that continuous self-mappings of compact convex

subsets of the function spaces $C^n[0,1]$ and $L_2[0,1]$ possess a fixed point. In attempting to generalize this result, J. Schauder [1] (1927) first extended it to metric topological linear spaces with a basis, and then, in 1930, established the following well-known and widely applied result:

<u>III.17. Theorem</u>: (Schauder [2]) Let $(X,|.|)$ be a normed linear space, and let C be a compact, convex subset of X. If f is a continuous self-mapping of C, then f has a fixed point in C.

<u>Proof</u>: Suppose f has no fixed point in C. Then by continuity of the function

$$F(x) = |f(x) - x|$$

on the compact set C, we have

$$\inf \{F(x) : x \varepsilon C\} = e > 0 \quad .$$

Since C is compact, there is a finite set $\{x_1,...,x_n\}$ of elements of C with the property that for any $x \varepsilon C$, there is an $x_i, i = 1,...,n$ such that

$$|x - x_i| < e \quad .$$

Let K be the convex hull of $\{x_1,...,x_n\}$ (see Remark II.4 (i)). We first show that there is a continuous mapping g from C into K such that

$$|g(x) - x| < e \quad , \qquad x \varepsilon C \quad .$$

To do this, define for each x in C and each $j = 1,\ldots,n$,

$$d_j(x) = e - |x - x_j| \qquad \text{if} \quad |x - x_j| < e \ ;$$
$$d_j(x) = 0 \qquad \text{if} \quad |x - x_j| \geq e \ .$$

It is clear that d_j is continuous and nonnegative, $j = 1,\ldots,n$, and that for any x in C there is at least one j for which $d_j(x) > 0$. Thus the mapping g defined by

$$g(x) = \sum_j d_j(x)x_j / \sum_i d_i(x) \quad , \quad x \in C \ ,$$

is a continuous mapping of C into K. Furthermore, for x in C we have

$$\begin{aligned} |g(x) - x| &= | \textstyle\sum d_j(x)x_j/\sum d_i(x) - x| \\ &= | \textstyle\sum d_j(x)(x_j-x)/\sum d_i(x)| \\ &\leq \textstyle\sum d_j(x)|x_j-x|/\sum d_i(x) \\ &< e \ , \end{aligned}$$

and g is a mapping with the desired property.

Now, since C is convex, K is contained in C, and hence the composite mapping gf^* is a continuous self-mapping of K, where f^* is the restriction of f to K. By the Brouwer fixed point theorem, gf^* has a fixed point x^* in K. For this point, however,

$$F(x^*) = |f(x^*) - x^*| = |f(x^*) - g(f(x^*))| < e \ ,$$

which contradicts the definition of e . Thus we conclude that f has a fixed point in C .

In a Banach space, Theorem III.17 has other, equivalent formulations. In order to state these and prove their equivalence, we require the following definitions and lemmas.

III.18. Definition: Let X and Y be normed linear spaces.

(i) A continuous mapping f from X to Y is said to be completely continuous if, for each bounded set $B \subset X$, there is a compact set $K \subset Y$ such that

$$f(B) \subset K \quad .$$

(ii) The closed convex hull of a set $K \subset X$ is defined to be the set

$$\operatorname{cco}(K) = \cap\{C \subset X : C \text{ is closed and convex, and } K \subset C\} \ .$$

(iii) The norm on X is said to be strictly convex (rotund), and X is called a strictly convex (rotund) space, if, for distinct x,y in X ,

$$|x| = |y| \leq 1 \quad \text{implies} \quad |(x+y)/2| < 1 \quad .$$

(Note that the 1 in the above definition may be replaced by any positive number.)

<u>III.19. Lemma</u>: If K is a compact set in a Banach space, then the closed convex hull of K, $\mathrm{cco}(K)$, is also compact.

<u>Proof</u>: This result was originally obtained by S. Mazur [1]; a proof is also given in Dunford and Schwartz [1], page 416.

<u>III.20. Lemma</u>: Let X be a normed linear space.

(i) If X is complete and separable, then there is a strictly convex norm on X which is equivalent to the original norm.

(ii) Suppose X is strictly convex, and $K \subset X$ is compact and convex. Then for each x in X, there is a unique point $P(x)$ of K for which

$$|x - P(x)| = \inf \{|x-y| : y \in K\} .$$

Furthermore, the mapping $P : X \to K$ so defined is continuous.

<u>Proof</u>: (i) See Clarkson [1], Theorem 9.

(ii) Let $x \in X$, and define h on K by

$$h(y) = |x-y| , \quad y \in K .$$

Then h is continuous on the compact set K, and hence there is a $z \in K$ such that

$$|x-z| = \inf \{|x-y| : y \in K\} .$$

If $x \in K$, then $z = x$ is uniquely determined. If $x \notin K$, let $z' \in K$ be such that $z' \neq z$ and

$$|x-z| = |x-z'| = d > 0 .$$

Since K is convex, $(z+z')/2 \in K$, so that

$$d \leq |x-(z+z')/2| .$$

But

$$|x-(z+z')/2| = |[(x-z) + (x-z')]/2| < d$$

by rotundity. This contradiction establishes uniqueness of $z = P(x)$.

To show continuity of P, let $\{x_n\}$ be a sequence in X which converges to $x \in X$. Since K is compact, the sequence $\{P(x_n)\}$ has a subsequence $\{P(x_{n_k})\}$ which converges, say to y, in K. Then we have

$$\lim_k |x_{n_k} - P(x_{n_k})| = |x-y| .$$

Furthermore, for each $n = 1,2,\ldots,$ the definition of P gives

$$|x-P(x)| \leq |x-P(x_n)| \leq |x-x_n| + |x_n-P(x_n)| ;$$

and similarly,

$$|x_n-P(x_n)| \leq |x_n-P(x)| \leq |x_n-x| + |x-P(x)| .$$

Thus,

$$||x_n-P(x_n)| - |x-P(x)|| \leq |x-x_n| \to 0 \quad \text{as} \quad n \to \infty .$$

Therefore, $|x-y| = |x-P(x)|$, so by uniqueness of $P(x)$, we have $y = P(x)$. Similarly, every subsequence of $\{P(x_n)\}$ has a subsequence which converges to $P(x)$. It follows that $\{P(x_n)\}$ converges to $P(x)$, and P is continuous.

III.21. Theorem: If X is a Banach space, the following statements are equivalent.

(i) A continuous self-mapping f of a compact convex set C in X has a fixed point in C.

(ii) A continuous self-mapping f of a closed convex set C in X has a fixed point in C if, in addition, $f(C)$ is contained in a compact subset K of C.

(iii) A completely continuous self-mapping f of X such that $f(X)$ is bounded has a fixed point.

Proof: (i) implies (ii): By Lemma III.19, $\mathrm{cco}(K)$ is compact; and since C is closed and convex, it follows that $\mathrm{cco}(K) \subset C$. Therefore,

$$f(\mathrm{cco}(K)) \subset f(C) \subset K \subset \mathrm{cco}(K) ,$$

and f is a continuous self-mapping of the compact convex set $\mathrm{cco}(K)$. Thus, (i) applies, so that f has a fixed point in $\mathrm{cco}(K)$, and hence in C.

<u>(ii) implies (iii)</u>: Let $C = cco(f(X))$; then C is closed, convex, and

$$f(C) \subset f(X) \subset cco(f(X)) = C \quad .$$

Furthermore, C is bounded since $f(X)$ is; hence by complete continuity of f, there is a compact set $K \subset X$ such that $f(C) \subset K$. The set $K \cap C$ is evidently a compact subset of C, and $f(C) \subset K \cap C$. Therefore, by (ii), f has a fixed point.

<u>(iii) implies (i)</u>: As a compact subset of the complete space X, C is itself a compact, complete normed linear space. Thus, C is separable, so by Lemma III.20 (i), we may assume the norm on C is strictly convex. Also, $f(C)$ is compact, so that $M = cco(f(C))$ is a compact convex subset of C. Let P be the mapping from C to M defined in Lemma III.20 (ii), and consider the mapping

$$f^* = fP : C \to M \quad .$$

Since M is compact, it is clear that f^* is completely continuous and $f^*(C)$ is bounded. Thus, by (iii), there is an x in C for which $f^*(x) = x$. However, since $x \in M$, $P(x) = x$, and we have

$$x = f^*(x) = f(P(x)) = f(x) \quad .$$

That is, x is a fixed point of f in C.

III.22. Remark: As noted by F. E. Browder [1], the Schauder fixed point theorem is included in the Lefschetz fixed point theorem for locally convex spaces (Lefschetz [6]). Browder further used the Lefschetz fixed point formula to obtain the following extensions of Schauder's theorem, in which the hypothesis on the mapping f is replaced by a corresponding hypothesis on one of its iterates:

Theorem A (Browder [1]): Let f be a completely continuous self-mapping of a Banach space X. If, for some integer $m > 0$, $f^m(X)$ is bounded, then f has a fixed point.

Theorem B (Browder [1]): Let $S_0 \subset S_1 \subset S$ be convex subsets of a Banach space X, with S_0 closed and S_1, S open; and let

$$f : S \to X$$

be continuous, with $f(S)^-$ compact. Suppose that for some integer $m > 0$, f^m is well-defined on S_1, while $f^m(S_1)$ is contained in S_0. Then f has a fixed point in S_0.

N. H. Anh [1] has recently shown that Theorem A remains valid if X is replaced by a closed convex subset of X.

Schauder also established a result corresponding to Theorem III.17, in which continuity of the function f is replaced by weak continuity, and compactness of the set C is replaced by weak compactness.

<u>III.23. Definition</u>: Let X be a normed linear space, and let X^* denote the conjugate space of X (i.e., X^* is the space of continuous linear functionals on X, with

$|A| = \sup\{|A(x)| : x \in X , |x| \leq 1\}$ for $A \in X^*$) .

(i) A sequence $\{x_n\}$ of elements of X is said to be <u>weakly convergent</u> to $x \in X$ if, for every A in X^*,

$$\lim_n A(x_n) = A(x) .$$

(ii) A subset C of X is <u>conditionally weakly sequentially compact</u> (hereafter abbreviated to cws-compact) if every sequence in C has a subsequence which converges weakly to a point of X.

(iii) A subset C of X is <u>weakly closed</u> if the weak limit of every weakly convergent sequence in C is also in C.

(iv) A subset C of X is <u>weakly compact</u> if C is weakly closed and cws-compact.

(v) A self-mapping f of X is <u>weakly continuous</u> if, whenever x_n converges weakly to x,

then $f(x_n)$ converges weakly to $f(x)$.

III.24. Lemma: Let X be a separable normed linear space. Then there is a sequence of functionals $\{A_n\}$ in X^* such that

(i) $|A_n| \leq 1$, $n = 1,2,\ldots,$ and

(ii) for any A in X^* with $|A| \leq 1$, there is a subsequence $\{A_{n_m}\}$ such that for each x in X , $A_{n_m}(x)$ converges to $A(x)$.

Proof: See Banach [2], page 232, Lemma 6.

III.25. Theorem (Schauder [2]) Let X be a separable Banach space, and let C be a convex, weakly compact subset of X . A weakly continuous self-mapping f of C has a fixed point in C .

Proof: Let the sequence $\{A_n\}$ in X^* be as in the Lemma, and let $\{e_i\}$ be a sequence of real numbers which converges to zero. For x in X , form the sequence

$$c_n(x) = e_n A_n(x) \quad , \quad n = 1,2,\ldots \; .$$

Clearly, for each x in X , $\{c_n(x)\}$ belongs to the space c_0 of null sequences (where we assume the norm on c_0 is given by $|\{a_n\}| = \sup_n |a_n|$) .

Define a mapping g from X to c_0 by

$$g(x) = \{c_n(x)\} \quad , \quad x \in X \quad .$$

The mapping g is evidently continuous and linear; it is also one-to-one, for if $g(x_1) = g(x_2)$, then $A_n(x_1) = A_n(x_2)$, $n = 1,2,\ldots$. This in turn implies $A(x_1) = A(x_2)$ for each A in X^* . But X^* is total over X ; i.e., $x_1 \neq x_2$ implies $A(x_1) \neq A(x_2)$ for some A in X^* (see, e.g., Wilansky [1], page 67, Corollary 3). Thus we conclude $x_1 = x_2$.

The set $C' = g(C)$ is convex, as may be seen from linearity of g and convexity of C . We now show C' to be sequentially compact, and hence compact. We note first that if $\{z_n\}$ is a sequence in C' , then $\{z_n\}$ is the image under g of a sequence $\{y_n\}$ in C , i.e., $z_n = g(y_n)$, $n = 1,2,\ldots$. Since C is weakly compact, $\{y_n\}$ contains a subsequence $\{y_{n_k}\}$ which converges weakly to an element y of C ; this subsequence defines a subsequence $\{z_{n_k}\}$ of $\{z_n\}$, where $z_{n_k} = g(y_{n_k})$. Consequently, to show sequential compactness of C' , it suffices to show that, for any sequence $\{x_n\}$ in C ,

$$x_n \to x \text{ weakly implies } g(x_n) \to g(x) \quad , \quad n \to \infty \; , \; x \in C \; .$$

Now, $x_n \to x$ weakly means $A(x_n) \to A(x)$ for all A in X^*, so in particular $A_i(x_n) \to A_i(x)$, $i = 1,2,\ldots$. Thus, the sequences

$$g(x_n) = \{e_1A_1(x_n) \quad , \quad e_2A_2(x_n),\ldots\} \quad , \; n = 1,2,\ldots$$

converge coordinate-wise to the sequence

$$g(x) = \{e_1A_1(x),\ e_2A_1(x),\ldots\} \quad .$$

Since the numbers e_i and $A_i(x_m)$ are bounded independently of i and m, the convergence is uniform, so that $g(x_n)$ converges to $g(x)$ in the norm of c_0. Consequently, C' is compact.

Consider now the mapping $h : C' \to C'$ defined by

$$h(g(x)) = g(f(x)) \quad , \quad x \in C \quad .$$

Then h is a continuous self-mapping of the convex compact set C', and hence by Theorem III.17, h has a fixed point y in C'. That is, $h(y) = y$ and $y = g(x)$ for some x in C. Thus, we have

$$g(x) = h(g(x)) = g(f(x)) \;;$$

the fact that g is one-to-one implies $f(x) = x$, and so x is a fixed point for f in C.

III.26. Remark: M. Krein and V. Smulian [1] have extended the result of Mazur (see Lemma III.19) to show that the closed convex hull of a cws-compact

set in a Banach space is weakly compact. Using this fact, they were able to give the following improved version of Schauder's theorem, Theorem III.25.

<u>Theorem</u> (Krein and Smulian [1]) Let C be a closed convex set in a Banach space X. If f is a weakly continuous self-mapping of C such that $f(C)$ is separable and cws-compact, then f has a fixed point in C.

<u>Proof</u>: Let $K = cco(f(C))$ (Definition III.18); by the above, K is weakly compact. Since C is closed and convex, $K \subset C$, and we have

$$f(K) \subset f(C) \subset K .$$

Furthermore, if N is a countable dense subset of $f(C)$, the set of convex linear combinations of N with rational coefficients is a countable dense subset of the convex hull of $f(C)$. Since the closure of a separable set is separable, it follows that K is separable, and hence can be regarded as a subset of a separable space. Thus, f and K satisfy the hypotheses of Theorem III.25, and f has a fixed point in K.

The Schauder fixed point theorem, Theorem III.17, was extended from normed linear spaces to linear topological spaces by A. Tychonoff [1]. We

preface the proof of Tychonoff's theorem with a definition and a lemma.

<u>III.27. Definition</u>: Let E be a linear space over a scalar field F (F is the set of real numbers or the set of complex numbers).

(i) E is a <u>topological linear space</u> if there is a topology defined on E such that the operations of addition and multiplication are continuous operations from $E \times E$ and $F \times E$, respectively, to E.

(ii) A topological linear space is said to be <u>locally convex</u> if every neighbourhood of the origin contains a convex neighbourhood of the origin.

<u>III.28. Lemma</u>: Let C be a compact subset of a Hausdorff topological linear space E. If $\{N\}$ is an open covering of C, then there is a finite open covering $\{U\}$ of C with the following property:

For each $U \in \{U\}$ there is an $N \in \{N\}$ such that $U \subset N$, and if $U' \in \{U\}$ intersects U, then also $U' \subset N$.

<u>Proof</u>: Since C is compact, C is covered by finitely many elements $N_1,\ldots,N_n$ of $\{N\}$. For each x in C, let $N^*(x)$ be a neighbourhood of x such that $\overline{N}^* \subset N_i$, for some $i = 1,\ldots,n$ ($\overline{N}^*$ denotes the closure of N^*). The collection $\{N^*\}$ is a covering of C, and hence has a finite subcovering, say $N_1^*,\ldots,N_m^*$.

Now for each x in C, define

$$O_1(x) = \cap \{N_k^* : x \in N_k^*\}$$

$$O_2(x) = \cap \{N_i : x \in N_i\}$$

$$O_3(x) = \cap \{E - \overline{N}_j^* : x \in E - \overline{N}_j^*\} .$$

Then the sets

$$U(x) = O_1(x) \cap O_2(x) \cap O_3(x) , \quad x \in C ,$$

form a finite open covering of C with the desired property. To show this, suppose $U(x)$, $U'(x')$ are two such sets, with $U \cap U'$ nonempty. Let N_k^* be one of the sets in the intersection defining $O_1(x)$. Then

$$U(x) \subset N_k^* \subset \overline{N}_k^* \subset N_i ,$$

for some $i = 1,\ldots,n$. The point x' is in $\overline{N}_k^*$, for otherwise, $x' \in E - \overline{N}_k^*$ implies $U'(x') \subset E - \overline{N}_k^*$, in which case U and U' are disjoint. Thus $x' \in \overline{N}_k^* \subset N_i$, so that N_i is one of the intersecting sets in $O_2(x')$, and we have $U' \subset N_i$.

III.29. Theorem (Tychonoff [1]): Let E be a topological linear space which is locally convex and Hausdorff. If C is a convex compact subset of E, then every continuous self-mapping f of C has a fixed point in C.

Proof: Suppose f has no fixed point in C. Then for all x in C, $f(x) \neq x$, and hence we may separate each $x \in C$ from its image $f(x)$ by disjoint open neighbourhoods $N'(x)$, $N(f(x))$, respectively. Furthermore, by continuity of f, for each x in C there is a neighbourhood $N(x)$ such that

$$N(x) \subset N'(x) \quad , \text{ and } \quad f(N(x)) \subset N(f(x)) \quad .$$

We thus obtain an open covering of C by neighbourhoods $N(x)$ which have the property that $N(x)$, $f(N(x))$ are disjoint. By local convexity of E, we may assume that each $N(x)$, $x \in C$, is convex.

Let $\{U_i : i = 1,\ldots,p\}$ be a finite open covering of C which is related to $\{N\}$ as in the lemma. By continuity of f, we may choose for each x in C a neighbourhood $V(x)$ such that

$$f(V(x)) \subset U_i \quad ,$$

for some $i = 1,\ldots,p$.

For each $i = 1,\ldots,p$, let x_i be a point of U_i, and let L be the convex hull of $\{x_1,\ldots,x_p\}$. Since the subspace generated by $x_1,\ldots,x_p$ is linearly homeomorphic to a finite-dimensional Euclidean space, it follows that L is homeomorphic to a Euclidean r-simplex, $r < p$. Consequently, L may be simplicially subdivided in such a way that each subsimplex is contained in some $V(x_i)$, with its image therefore contained in some U_k, $i,k = 1,\ldots,p$.

Let $\{y_i\}$ be the set of vertices of these subsimplexes, and define

$$g(y_i) = x_k \quad ,$$

where x_k is determined by the relation

$$f(y_i) \in U_k \quad .$$

(If there are several values of k for which the above holds, pick any one of them.) Extending the definition of g linearly to any element of L, we obtain a continuous self-mapping of L. By Brouwer's theorem, g has a fixed point, say x^*, in L.

Now, let x be any point in L; x is contained in some subsimplex, say $s = [y_0,\ldots,y_r]$. Since $f(s) \subset U_k$ for some k, we have $f(x)$,

$f(y_i) \in U_k$, $i = 0,1,\ldots,r$. Furthermore, by definition of g , $g(y_i)$ and $f(y_i)$ are contained in the same U_{k_i} , $i = 0,1,\ldots,r$. Thus, $f(y_i) \in U_k \cap U_{k_i}$; and if N is a member of the covering $\{N(x) : x \in C\}$ which contains U_k , then N also contains each U_{k_i} , $i = 0,\ldots,r$. Therefore each $g(y_i)$ belongs to N , and hence by convexity of N , $g(x) \in N$. Thus, for any $x \in L$, we have established that $f(x)$ and $g(x)$ belong to the same member of the covering $\{N(x)\}$. In particular, $x^* = g(x^*)$ and $f(x^*)$ belong to the same element of $\{N(x)\}$. This, however, contradicts the fact that each $N(x)$ is disjoint from its image $f(N(x))$, and the theorem is proved.

The above proof is the original one of Tychonoff. Alternate proofs, of a more analytical nature, can be found in Edwards [1], Chapter 3 , and Bonsall [1], Chapter 3 . A brief proof, which uses the theory of topological semifields, has been given by T. A. Sarymsacov [1].

<u>III.30. Remarks</u>: (i) The corresponding extension of Theorem III.21, (ii) to locally convex linear topological

spaces has been given by M. Hukuhara; moveover, in Hukuhara's theorem, the set C is assumed to be merely convex:

<u>Theorem</u> (Hukuhara [1]) Let E be a locally convex Hausdorff topological linear space. If C is a convex set in E, and if f is a continuous self-mapping of C such that

$$f(C) \subset A \subset C \quad ,$$

for some compact set A, then f has a fixed point in C.

(ii) It does not appear to be known whether Tychonoff's theorem holds in an arbitrary topological linear space (without the assumption of local convexity), even if the space is metrizable. A proof by Lefschetz [7, page 119] that a continuous self-mapping of a compact convex subset of a metric linear space possesses a fixed point, has been shown by V. Klee [1] to be erroneous.

CHAPTER IV

APPLICATIONS OF FIXED-POINT THEOREMS

The continuing interest in fixed-point theorems is due largely to their wide applicability, particularly in establishing the existence of solutions to differential, integral, and other functional equations. In this chapter we present some examples of applications of fixed point theorems.

The first example, due to B. H. Arnold [1], is a proof of the Fundamental Theorem of Algebra which makes use of Brouwer's fixed-point theorem:

<u>Theorem</u>: Let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a complex polynomial; then then equation

$$f(z) = 0$$

has a root.

<u>Proof</u>: For $z = re^{i\Theta}$, $0 \leq \Theta \leq 2\pi$, and $R = 2 + |a_1| + \ldots + |a_n|$, define

$$g(z) = z - f(z)/Re^{i(n-1)\theta r} \quad , \quad |z| \leq 1$$

$$g(z) = z - f(z)/Rz^{n-1} \quad , \quad |z| \geq 1 \ .$$

Then g is continuous and well-defined for all z , since neither denominator vanishes, and for

$|z| = 1$, the two expressions coincide. Furthermore, g is a self-mapping of the set

$$C = \{z : |z| \leq R\} .$$

To see this, we note that for $|z| \leq 1$,

$$|g(z)| \leq |z| + |f(z)/R| \leq 1 + (1+|a_1|+ \ldots +|a_n|)/R \leq 2 \leq R;$$

and for $1 \leq |z| \leq R$,

$$\begin{aligned} |g(z)| &= |z - z/R - (a_1+a_2z^{-1} + \ldots + a_nz^{1-n})/R| \\ &\leq |(R - 1)z/R| + (|a_1|+ \ldots +|a_n|)/R \\ &\leq R - 1 + (R - 2)/R = R - 2/R \\ &< R . \end{aligned}$$

Thus, by the Brouwer fixed-point theorem, there is a z^* in C such that $g(z^*) = z^*$. That is, we have either

$$z^* - f(z^*)/Re^{i(n-1)\theta r} = z^* ,$$

or

$$z^* - f(z^*)/R(z^*)^{n-1} = z^* ,$$

depending on whether $|z^*| \leq 1$ or $|z^*| \geq 1$. In either case, it follows that $f(z^*) = 0$, which proves the theorem.

For further examples, we consider the initial value problem

(1) $\qquad x' = f(t,x) , \qquad x(t_o) = x_o .$

Generally speaking, the Contraction Mapping Theorem can

be used to prove the existence of a unique solution for (1) if the function f satisfies a Lipschitz condition in x. On the other hand, this condition can be relaxed and stronger fixed-point theorems (e.g., the Schauder theorem) used to establish the existence of a solution; this is done, however, at the expense of uniqueness. The details follow.

<u>Theorem</u>: Let $t_0 \in E_1$, $x_0 \in E_n$, and let D be the parallelepiped:

$$D = \{(t,x) \in E_1 \times E_n : |t - t_0| \leq a^* , |x - x_0| \leq b\} .$$

Further suppose that $f : D \to E_n$ is continuous and satisfies a Lipschitz condition in x; i.e., for some $K > 0$,

$$|f(t,x_1) - f(t,x_2)| \leq K|x_1 - x_2|$$

whenever (t,x_1), (t,x_2) belong to D. Then there is a unique function $x : [t_0 - a, t_0 + a] \to E_n$ such that

$$(1) \qquad x' = f(t,x) , \qquad x(t_0) = x_0 .$$

Here, $a = \min\{a^*, b/M\}$, where $M = \max\{|f(t,x)|:(t,x) \in D\}$.

<u>Proof</u>: The differential equation (1) is equivalent to the integral equation

$$(2) \qquad x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds ,$$

so that it suffices to show the existence of a unique solution for (2).

Let C be the space of continuous functions $x(t)$ from $[t_0-a, t_0+a]$ to E_n, with metric

$$d(x,y) = \max\{|x(t)-y(t)| : t \in [t_0-a,\ t_0+a]\} .$$

Then (C,d) is complete, so that the closed subset

$$C^* = \{x \in C : |x(t)-x_0| \leq b\}$$

is also a complete metric space. For each $x \in C^*$ define a mapping $T(x)$ by

$$T(x)(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \quad , \ t \in [t_0-a, t_0+a] .$$

The mapping $T : C^* \to C$ is a self-mapping of C^*, since, for $t \in [t_0-a, t_0+a]$,

$$|T(x)(t) - x_0| = |\int_{t_0}^{t} f(s,x(s))\, ds| \leq M|t-t_0| \leq Ma \leq b .$$

Furthermore, T satisfies

$$|T^n(x)(t) - T^n(y)(t) < (K|t-t_0|)^n\, d(x,y)/n!, \ n = 1,2,...,$$

where $x,y \in C^*$, $t \in [t_0-a, t_0+a]$, as is now shown by induction on n.

For $n = 1$, we have

$$|T(x)(t) - T(y)(t)| \leq \int_{t_0}^{t} |f(s,x(s)) - f(s,y(s))||ds|$$

$$\leq K|t-t_0| \max |x(s) - y(s)|$$

$$= K\ |t-t_0|d(x,y) \quad .$$

Next, suppose the inequality holds for some $m \geq 1$; then

$$|T^{m+1}(x)(t)-T^{m+1}(y)(t)| = |\int_{t_0}^{t} [f(s,T^m(x)(s)) - f(s,T^m(y)(s))]ds|$$

$$\leq K\int_{t_0}^{t} |T^m(x)(s) - T^m(y)(s)||ds|$$

$$\leq (K^{m+1} d(x,y)/m!) \int_{t_0}^{t} |s-t_0|^m|ds|$$

$$= (K|t-t_0|)^{m+1} d(x,y)/(m+1)! \quad ,$$

and the induction is complete.

Thus since $|t - t_0| \leq a$, we have $d(T^n(x),T^n(y)) \leq ((Ka)^n/n!) \ d(x,y)$, $n = 1,2,...,$ and by the Contraction Mapping Theorem (see Remark I.3 (iv), pages 4-5), T has a unique fixed point in C^* . That is, there is a unique function $x \varepsilon C^*$ such that

$$x(t) = T(x)(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\, ds \quad ,$$

and the theorem is proved.

<u>Theorem</u>: Let T_1 be a closed interval in E_1 , let $y_0 \varepsilon E_n$, and for $r > 0$, let $B = \{y \varepsilon E_n : |y - y_0| \leq r\}$. Further, suppose

$f : T_1 \times B \to E_n$ is continuous and bounded, with $|f(t,y)| \leq M$, for $t \in T_1$, $y \in B$. For $t_0 \in T_1$ and $c = r/M$, let

$$T = T_1 \cap [t_0 - c \, , \, t_0 + c] \quad .$$

Then, there is a continuous function $x : T \to B$ such that

$$(1') \quad x' = f(t,x) \quad , \quad x(t_0) = y_0 \quad .$$

Proof: As in the proof of the preceding theorem, it suffices to show the existence of a solution for the integral equation

$$(2') \quad x(t) = y_0 + \int_{t_0}^{t} f(s,x(s))ds \quad .$$

Let E be the set of continuous functions $x : T \to E_n$; with addition and scalar multiplication defined pointwise, and with the norm defined by

$$|x| = \sup \{|x(t)| : t \in T\} ,$$

E is a Banach space. Consider the set $A = \{x \in E : |x - y_0| \leq r\}$, where $y_0(t) \equiv y_0$. A is clearly a closed and convex subset of E. For each $x \in A$, define a mapping $u(x)$ by

$$u(x)(t) = y_0 + \int_{t_0}^{t} f(s,x(s))ds, \quad t \in T \quad .$$

The mapping $u : A \to E$ is a self-mapping of A, since if $x \in A$, we have

$$|u(x)(t) - y_0(t)| = |\int_{t_0}^{t} f(s,x(s))ds|$$
$$\leq M|t - t_0| \leq Mc = r \quad ,$$

and hence $|u(x) - y_0| \leq r$.

Next, to show continuity of u on A , let $\{x_n\}$ be a sequence in A with $x_n \to x \in A$. For $s \in T$, let $g_n(s) = f(s,x_n(s))$, $n = 1,2,\ldots$, and let $g(s) = f(s,x(s))$. Since f is uniformly continuous on $T \times B$, it follows that $g_n \to g$ uniformly on T , and hence $u(x_n)(t) \to u(x)(t)$ uniformly on T . Consequently, $u(x_n) \to u(x)$, and u is continuous.

To apply the Schauder fixed-point theorem (Theorem III.21 (ii)), it remains to show that $u(A)$ has compact closure. This, however, follows immediately from the Arzela-Ascoli Theorem (see Dunford and Schwartz [1], page 266), by noting that $u(A)$ is bounded, since $u(A) \subset A$, and $u(A)$ is equicontinuous, since

$$|u(x)(t) - u(x)(t')| = |\int_{t'}^{t} f(s,x(s))ds| \leq M|t - t'| \quad ,$$

for $x \in A$, $t,t' \in T$.

Thus u has a fixed point x^* in A :

$$x^*(t) = u(x^*)(t) = y_0 + \int_{t_0}^{t} f(s,x^*(s))ds \quad ,$$

and x^* is a solution of (2) .

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